Some Applications Of Quantum Field Theoretic Dualities To Superconducting Systems

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by

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Dedicated to my parents...

Krishna Kali Mukherjee and Nita Mukherjee

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Abstract

The main theme of this thesis is the application of a field theoretic duality to superconducting systems. In theoretical physics duality is such a concept by which one can represent a system by two (or more) different set of field theoretic variables. One of these dualities is the boson-vortex duality which maps vortices existing in a 3+1 dimensional theory to strings and in 2+1 dimension it maps vortices to point particles (Particle-vortex duality). In the existing literature this duality is applied to systems described by scalar fields or systems with static spins. Here in this thesis we have applied this duality to systems where both charged bosonic and fermionic fields are present. Such a model was originally proposed by Friedberg and Lee as a model for high T_c cuprate superconductors and was later adopted in theoretic models of pseudogap, BCS-BEC crossover scenarios, emergent supersymmetry in condensed matter systems, etc.

We have used a model similar to Friedberg-Lee model but relativistic in nature. These bosonic and fermionic fields in our model have a global $U(1)$ charge and therefore interact with electromagnetic gauge field. Through our investigation into these systems we have found a few interesting features of such models. The boson-vortex duality, when applied to 3+1 dimensional Abelian Higgs model (AHM), leads to a dual theory where vortices are represented by 3+1 dimensional strings and these strings interact via a 2-form gauge field. We have applied this duality to the boson fermion mixture model and have shown that the fermions in the dual theory couple to 2-form gauge potential and thus interact with the dual strings via the exchange of this 2-form gauge potential. We have named this interaction emergent electron-vortex interaction. In this thesis we have mainly analyzed some consequences of such an interaction.

One of the main consequences of this electron-vortex interaction is the attachment of vortices to electrons. The main reason for such an attachment in 3+1 dimensional theory is the electron's non-zero spin magnetic moment. For this reason we have called this phenomenon spin-flux attachment. We have also presented some theoretical arguments for the pair formation of these vortex attached fermions. It was argued that two such vortex attached fermions, when coming closer to each other, may become connected to each other by two vortex lines. One can also calculate the interaction potential and show that it increases linearly with the distance between two electrons when the exchange momentum is large compared to the mass of the 2-form field. This linear potential also suggests that the electrons are connected by vortices or effective strings. However, when the two length scales of the theory- the inverse scalar mass and the inverse photon mass-becomes comparable, the mechanism described above fails. We have found arguments for fermion pairing in this case as well. The dual theory of vortex lines can be generalized in terms of a string field theory. In the false vacuum of the string field the mass of the 2-form vanishes. In such a situation the electron-electron effective potential becomes linear in the interparticle distance which leads to stable pairs of fermions.

Another consequence of this flux attachment phenomenon is the statistical transmutation of the flux attached fermions. To show this we have reduced the 3+1 dimensional dual theory to 2+1 dimensions by assuming that all the particles and their interaction are confined to a 2+1 dimensional plane. We have further shown that reduced 2+1 dimensional theory may lead to the same flux attachment equation obtained previously but now the coupling constants are modified by a new factor. It is shown that due to this flux attachment equation when two non-relativistic vortex attached particles are moved around each other an Aharonov-Bohm phase is induced in their wave function which depends

on the new constant factor introduced in the process of dimensional reduction. If this factor varies arbitrarily this phase can also vary and become different from 0 or π. This arbitrariness of this phase factor may be an indication of the statistical transmutation of such flux attached particles.

In this way, our investigation, via the application of boson-vortex duality, has brought out some unobserved features of the boson-fermion system which may draw the attention of the scientific community.

সংক্ষিপ্তসার

আমার গবেষণার মূল বিষয়বস্তু হল কণাধর্মীয় (Quantum) ক্ষেত্রতাত্ত্বিক দ্বিবিধত্ব (Field theoretic duality) এবং অতিপরিবাহী পদার্থের ক্ষেত্রে এর প্রয়োগ। তাত্ত্বিক পদার্থবিদ্যায় দ্বিবিধত্ব হল এমন একটি ধারনা যার মাধ্যমে একটি সিস্টেম কে দুটি ভিন্ন ক্ষেত্রতাত্ত্বিক চলরাশি সমূহের দ্বারা প্রকাশ করা সম্ভব। এইরকম বিভিন্ন দ্বিবিধত্বের মধ্যে একটি হল বোসন – আবর্ত (Vortex) দ্বিবিধত্ব (বা কণা - আবর্ত দ্বিবিধত্ব) যার মাধ্যামে একপ্রকার আবর্তের তত্ত্বকে স্ট্রিং তত্ত্বরূপে (বা কণার তত্ত্বরূপে) প্রকাশ করা সম্ভব। এখানে আবর্ত হল একধরনের সমষ্টিগত উদ্দীপনা (Collective Excitation) যা টাইপ-II অতিপরিবাহীর ক্ষেত্রে দেখা যায়। সাধারণত এই দ্বিবিধত্ব বোসন ক্ষেত্র দ্বারা বর্ণিত সিস্টেমের জন্য প্রয়োগ করা হয়। আমার গবেষণায় এই দ্বিবিধত্ব প্রয়োগ করা হয়েছে এমন এক ধরনের সিস্টেমের ক্ষেত্রে যেখানে বোসন কণা ছাড়াও ফের্মিয়ন কণাও উপস্থিত থাকে। অতীতে এই ধরনের সিস্টেমের প্রথম অবতারণা করা হয়েছিল কিউপ্রেট অতিপরিবাহীর ক্ষেত্রে। যদিও পরবর্তীকালে অনুরূপ কিছু মডেল প্রয়োগ করা হয়েছে সিউডোগ্যাপ এর সম্ভাব্য ব্যাখ্যায়, বিসিএস – বিইসি ক্রসওভার, উদ্ভূত সুপার সিমেট্রি (Emergent Supersymmetry) ইত্যাদি ক্ষেত্রে।

আমার কাজেও এইধরনেরই একটি মডেল ব্যবহার করা হয়েছে যেখানে বোসন ও ফের্মিয়ন কণাদের যথাক্রমে ক্লেইন – গর্ডন অ্যাকশন এবং ডিরাক অ্যাকশন দ্বারা বর্ণনা করা হয়েছে। এখানে এই দুই ধরনের কণাই তড়িতাহিত এবং এরা তড়িৎ - চুম্বকীয় ক্ষেত্রের (Electromagnetic Field) মাধ্যমে মিথস্ক্রিয়া করে। আমার গবেষণার মাধ্যমে এই মডেলের ক্ষেত্রে কিছু নতুন সম্ভাব্য ঘটনার আভাস পাওয়া গেছে। বোসন – আবর্ত দ্বিবিধত্ব প্রয়োগের মাধ্যমে এটি দেখান যায় যে চতুর্মাত্রিক (3+1 dimensional) অ্যাবেলিয়ান হিগস্ মডেল টি আবর্তের উপস্থিতিতে একধরনের স্ট্রিং তত্ত্বের সাথে সমতুল্য। এখানে স্ট্রিংগুলি আসলে চতুর্মাত্রিক আবর্ত গুলিকেই চিত্রিত করে এবং ২- ফর্ম ক্ষেত্রের মাধ্যমে মিথস্ক্রিয়া করে। আমরা বোসন ফের্মিয়ন মিশ্রণ এর জন্য এই বোসন - আবর্ত দ্বিবিধত্ব প্রয়োগ করে দেখিয়েছি যে দ্বৈত তত্ত্বে তড়িতাহিত ফের্মিয়ন কণাগুলি ২-ফর্ম ক্ষেত্রের মাধ্যমে নিজেদের সাথে এবং আবর্ত গুলির সাথে মিথস্ক্রিয়া করে। এই ধরনের মিথস্ক্রিয়া প্রথম আমাদের গবেষণাতেই দেখানো গেছে এবং নাম দেওয়া হয়েছে উদ্ভূত ফের্মিয়ন - আবর্ত মিথস্ক্রিয়া (Emergent fermion-vortex interaction)। আমার গবেষণামূলক প্রবন্ধে মূলত এই মিথস্ক্রিয়ার প্রভাব নিয়ে আলোচনা করা হয়েছে।

এই ফের্মিয়ন - আবর্ত মিথস্ক্রিয়ার একটি প্রধান প্রভাব হল ফের্মিয়ন ও আবর্তের সংযুক্তিকরণ। এর প্রধান কারণ হল এই যে, ফের্মিয়ন গুলি তাদের স্পিন ধর্মের জন্য সূক্ষ্ম চুম্বকের মত আচরণ করে। এই চুম্বকীয় ধর্মের জন্য ফের্মিয়ন গুলি প্রায় স্থির অবস্থাতেও আশেপাশের তড়িতাহিত বোসন ক্ষেত্রের মধ্যে আবর্ততা (Vorticity) সৃষ্টি করে। আমরা এই প্রবন্ধে এই ধরনের সংযুক্তি কে স্পিন – আবর্ত সংযুক্তি বলেছি। এই প্রবন্ধে আবর্ত সংযুক্ত ফের্মিয়ন গুলি কিভাবে একটি ফের্মিয়ন যুগ্মের (Fermion pair) সৃষ্টি করতে পারে তার স্বপক্ষেও তাত্ত্বিক যুক্তি প্রণয়ন করা হয়েছে। আবর্তের সাথে সংযুক্তি ধর্মের জন্য দুটি ফের্মিয়ন দুটি চতুর্মাত্রিক আবর্তের সাথে সংযুক্ত হয়ে একটি ফের্মিয়ন যুগ্মের সৃষ্টি করতে পারে। আমরা এটাও দেখিয়েছি যে, এই ফের্মিয়ন গুলির মিথস্ক্রিয়ার বিভব (Interaction potential) পারস্পরিক দূরত্বের সমানুপাতিক হতে পারে। এই ধরনের বিভবও এটাই নির্দেশ করে যে ফের্মিয়ন গুলি একে অপরের সাথে আবর্ত দ্বারা বা কার্যকরী স্ট্রিং দিয়ে সংযুক্ত।

আমরা ফের্মিয়ন - আবর্ত সংযুক্তির আরেকটি উল্লেখযোগ্য প্রভাব দেখাতে পেরেছি যেটা হল এই আবর্ত সংযুক্ত ফের্মিয়ন গুলির পরিসংখ্যানগত রূপান্তর (Statistical transmutation)। আমরা দেখিয়েছি যে যখন এই ফের্মিয়ন কণা গুলি ত্রিমাত্রিক স্থান-কালে আবদ্ধ থাকে, তখন তারা ফের্মি - ডিরাক পরিসংখ্যানের পরিবর্তে বোসন ও ফের্মিয়ন আচরণের মধ্যবর্তী একধরনের আচরণ দেখায় যাকে এনিয়ন পরিসংখ্যান বলে।

এভাবে আমাদের ফের্মিয়ন – আবর্ত মিথস্ক্রিয়া সংক্রান্ত গবেষণা বোসন – ফের্মিয়ন মিশ্রণ সম্বন্ধিত অনেক নতুন দিক উদ্ঘাটন করেছে, যা এই সিস্টেম সম্পর্কে গবেষক মহলের নতুন ভাবনা - চিন্তার উদ্রেক ঘটাতে পারে।

List of Publications

The thesis is based on the followings papers

1. **Shantonu Mukherjee**, Amitabha Lahiri, *"Emergent vortex–electron interaction from dualization"*, **Annals Phys. 418 (2020) 168167.**

2. **Shantonu Mukherjee**, Amitabha Lahiri, *"Spin gauge theory, duality and fermion pairing"*, **JHEP 02 (2022) 068.**

3. **Shantonu Mukherjee**, Amitabha Lahiri, *"Spin-flux attachment by dimensional reduction of vortices"*, **Nucl.Phys.B 986 (2023) 116050.**

Contents

Chapter 1

Introduction

From very early days of quantum mechanics, scientists are successfully applying quantum principles to many body systems to understand their macroscopic properties observed in different experiments. Such explorations started from explaining spectra of atoms and molecules and latter extended to the study of materials in different phases and also to the exploration of subatomic world. Based on the length scale which needs to be probed (or equivalently energy range of the probing agent) many body physics can be roughly divided into two broad categories. One is the physics of solids, fluids (gas or liquid) in which the constituent particles (atoms or molecules or electrons) have very restricted motion and is called condensed matter physics (or low energy physics). In this case collective behaviour of electrons in the background of periodic array of atoms or molecules become important. Another is the physics of elementary particles which governs the behaviour of the nucleus and its constituent particles. At this energy range particles are essentially relativistic and thus capable of producing many other particles of different types. However, the physics of plasma in stars, quark-gluon plasma realized in particle accelerators, cosmic ray shower would also fall into this category. The physics of such systems is in general referred to as high energy physics.

To deal with the physics of manybody systems, quantum principles, which were initially proposed for one body physics, were extended to many body field theories where the technology of second quantization becomes essential. In a second quantized theory the many body states are the vectors of Fock space which is the space of states characterized by the occupation numbers. Thus such construction would allow states with zero occupancy to infinitely high occupancy. The operators in such a construction are represented in terms of creation or annihilation operators which can act on the states of Fock space. Although the technique of second quantization were used both in non-relativistic or relativistic many body theories their subsequent developement took different paths because in the latter case relativistic symmetries are essential whereas in the former case relativistic symmetry is absent. Despite of this difference these two fields keep influencing each other and we shall see in latter discussions how ideas developed in one case becomes essential in explaining various features in the other case.

Historically it was Albert Einstein who, immediately after the developement of the idea of quantum nature of light, applied the idea of photon to lattice vibrations. His idea of modeling a solid as a gas of phonons was later extended by Debye to explain the specific heat of solids. Thus phonon was the first ever postulated collective excitation of a solid. Later on such ideas were extended for various phases of matter like magnons in ferromagnets, plasmons in charged plasma, polarons in a ionic matter, composite fermions in fractional quantum Hall state, Bogoliubov quasi-particles in superconductors etc. Apart from them there are collective excitations called solitons which are being discovered in different systems of current interest. Example of them include vortices in type-II superconductors, magnetic skyrmions in different magnetic systems, magnetic monopoles in spin ice etc.

The idea of spontaneous breakdown of a global symmetry was developed in the context of ferromagnetism where below Curie temperature the phase of the spins in a large region takes a unique value and hence break rotational symmetry ($SO(3)$). Following the behaviour of π meson Nambu first postulated that π meson could be the soft modes (Goldstone modes) associated with the spontaneous breakdown of some underlying symmetry and gave some idea of what the underlying symmetry could be. [1]

The ideas of renormalization group (RG group), which were developed to describe the singular behaviour different physical quantities near the phase transition point (Critical phenomenon), were later proved to be useful in developement of the idea of asymptotic freedom which is essential in QCD physics. These ideas were also useful in development of the ideas of unification of forces.

The technology of diagramatic perturbation theory and Green's function which were initially developed in the context of relativistic field theories (elementary particle physics and nuclear physics) were adopted in modern condensed matter contexts as well. In particular they are used in fermi liquid theory and in the subsequent developement of BCS theory of superconductivity. The effective description of superconductivity in terms of Ginzberg-Landau theory can be thought of as the material realization of Higgs mechanism by which photon gets mass.

Recently such interdisciplinary research has got a boost as new phases of matter are being discovered in low temperature experiments. A bunch of new ideas started to grow in the early 80's when physics of lower dimensional systems were being probed in experiments. In particular the discovery of integer quantum Hall effect (IQHE) [2] and subsequent discovery of fractional quantum Hall effect(FQHE) [3] found in two dimensional electron gas (2DEG) under high magnetic field and low temperature marks a new era for quantum condensed matter physics. Although the phenomenology of IQHE can be explained from the Landau level physics for non-interacting electrons, explanation of FQHE would require consideration of inter-electron interaction. In particular it was shown by various authors that the effective theory, which can explain the plateaus in the resistivity curve, incorporate a new breed of quasi particles called anyons (or composite fermions [4]. These anyons was first thought to exist in planar theories by Leinass and Myrheim [5]. Immediately after their work Wilczek showed that composites of electrons and vortex like extended objects can obey fractional statistics and named them anyons [6]. Anyons are incorporated into the effective description of FQHE via a gauge theory known as Chern-Simons(CS) theory. This Chern-Simons theory and various other theories that we are going to discuss are classified as topological field theories(TQFT). These TQFTs appear in effective description of different phases of matter and thus establishes a connection to relativistic field theories with low energy condensed matter systems.

Another important discovery of that decade is that of cuprate superconductors which shows a transition temperature as high as Nitrogen liquefaction temperature and is thus very important for practical purposes as well. Experiments suggests that electrons in CuO plane in cuprates are responsible for superconductivity. This led to the developement of the semionic theory [7] for cuprates which again incorporates the idea of anyons and hence Chern-Simons gauge theory. However this and various other theoretical attempts have failed to give a complete varifiable theory for the rich phenomenology of cuprates. In spite of this failure such theoretical research has given rise to concepts like spin liquids [8], non-fermi liquids [9], strange metals [10] etc which demand to be independent subjects of research on their own right.

Theoretical research on quantum hall effects and related phenomenon have given rise to a new subject called topological phases of matter [11, 12] which is at the heart of modern developments of condensed matter research. The central theme of this topic is the classification of states of matter by the topology of the eigenstates of the Hamiltonian governing the dynamics of electrons in the the system in consideration. A famous example of such topological matter is topological insulators which are characterized by the gapped spectrum in the bulk and gapless edge states which carry the current. For a d-dimensional insulator these gapless states can be described by $d-1$ dimensional mass less Dirac equation [12]. This again draws connection of topological matter to relativistic physics. Furthermore, the electromagnetic response of such materials can be summarized by a TQFT which is called an "axion term" which gives rise to a Chern Simons term in the edge. Other examples would be Dirac/Weyl semimetals which are characterized by appearance of massless Dirac/Weyl particles close to Dirac/Weyl node of the Hamiltonian [13]. Such materials can be useful for table top experiments for various phenomenon predicted to exist in relativistic physics. As example, particles like spin-3/2 fermions or Rarita-Schwinger fermions [14] which are not found in natural world, but predicted in the context of elementary particle physics, are also shown to exist theoretically in certain similar materials at low energy [15–17]. Similarly, Majorana fermions which were never found naturally were predicted to exist at the vortex core in a p-wave superconductors placed in proximity to a TI [18]. These particles are also shown to be useful for topologically protected quantum computation [19, 20] and it have got huge attention from both theoretical and experimental condensed matter community.

Apart from the theory of topological condensed matter, the principles of topology are also important in definition of topological solitons which plays an important role in non-perturbative sectors of particle physics (e.g. QCD) as well as in low temperature phases of condensed matter physics. Topological solitons are solutions of classical field theories with specific asymptotic conditions on the fields which constrain the fields to be non-zero very close to the origin which in turn localizes the solutions. Thus these solutions acts as localized objects or particles. Among various topological soliton solutions the first ever experimentally detected soliton was a vortex in type-II superconductor and they are also the central theme of this thesis. Vortex in any fluid is characterized by non zero vorticity which gives a measure of rotation of the fluid and is defined as the curl of velocity vector. The properties of topological vortices is that their vorticity is quantized i.e. an integer multiple of vorticity quantum 2π. This integer is called a winding number and comes from the homotopic mapping involved in their definition [21]. This non-zero winding number expresses the fact that the vortices are topologically stable and hence dissipation of a vortex from a non-zero winding say $n = 1$ to $n = 0$ needs infinite energy and is not allowed. We shall discuss the role of topology in the definition of the vortices in the next part of this chapter. At this point we would like to mention that apart from vortices there are other solitons like magnetic monopoles in spin ice material [22], skyrmions in magnetic materials [23] which are observed in experiments [24, 25] and plays important roles in explaining phenomenological features of such systems.

In our work we shall be interested in systems where fermions are also present in addition to vortices in a background scalar (Bosonic) matter. The vortex solutions also involve a particular configuration of the electromagnetic field due to which a non zero magnetic flux is trapped inside the vortex configuration. Now, if we keep charged fermions in the system where vortex configuration already exists, the fermions would couple to the electromagnetic field produced by vortices. In such a case one may write down the Dirac equation coupled to electromagnetic field A_μ of vortices like the following

$$\left(i\gamma^\mu \partial_\mu - m - e\gamma^\mu A_\mu\right)\psi = 0, \tag{1.1}$$

where $A_\phi \to \frac{c}{er}\sqrt{\frac{\pi|\phi|}{2er}}e^{-e|\phi|r}$ which is produced by a vortex. Thus it is quite expected that in a system where both fermions and vortex are present an interaction among those is inevitable. Studies of such systems was also considered by other authors in which they have considered Dirac equation in the electromagnetic vector potential produced by vortices and tried to solve them. Such solutions indicate presence of zero modes of fermionic field trapped inside vortex core [26, 27]. In particular similar studies where zero modes describe Majorana fermions [28] have gained considerable high attention due to their possible application in quantum computation as we mentioned earlier. However, such studies consider vortices as static configuration of scalar field as well as electromagnetic field and hence neglect their dynamics. We shall consider this question in our thesis and try incorporate their dynamics and their interaction by invoking the idea of duality into our work.

The subject of duality is itself a leading research topic in high energy physics as well as in condensed matter theory. Dual transformation allows one to describe a system in different field variables which define two theories whose parameters (temperature, couplings etc.) are related. Among different broad classes of dualities strong-weak duality is of particular interest which maps a strongly interacting system to a weakly interacting one and thus makes it possible to apply perturbation techniques. In the context of high energy physics (HEP) such dualities become essential to analyse non-perturbative properties of non-Abelian gauge theories [29–31], while in the context of condensed matter strong-weak dualities are applied to strongly correlated systems like spin liquids, strange metal phase of Cuprate superconductor, Mott insulators, Luttinger liquids etc [32, 33]. Thus dualities have become a common language to both community.

The duality in which we are interested in is called boson-vortex duality or a particle vortex duality. It was shown previously that when applied to a 2 dimensional spin system, this duality maps the vortices, that may appear in the system, to point particles whose current is the vorticity in the original theory and they interact via a new emergent gauge field. We shall review this duality in quite details. As we shall see this gauge interaction gives rise to a logerithmic potential between the point vortices which suggests that the vortices are confined. There exists a phase transition from this confined phase to a deconfined phase where isolated vortices are allowed to appear. This phase transition is known as BKT transition and is fundamentally different from other known phase transitions studied in the literature. Experimental realisation of this predictions was awarded with Nobel prize in recent past. However, higher dimensional generalisation of this duality is of interest to HEP community as well because such a higher dimensional duality maps the 3+1 dimensional vortices to effective strings in the dual theory. Thus the dual theory is an effective string theory where strings interact via a 2-form gauge field. The dynamics of these effective strings, expressed by Nambu-Goto term, also gives the dynamics of the vortex lines in the original system. Thus it is evident that the 3+1 dimensional duality would help us include the dynamics of vortices in terms of effective strings into our study of interaction of fermions with vortices. However, the construction of the duality in our case would differ from the standard cases because we include fermions into the system. As we shall see the dual Lagrangian in our case contains an electron-vortex gauge interaction as well as vortex-vortex gauge interaction. We shall focus on the this electron-vortex interaction throughout this thesis and shall explore it's various consequences. In particular, we shall see how such an interaction may lead to an electron-vortex attachment which in turn leads to pairing of fermions. Such a novel type of pairing may be realized in some appropriate system. In a lower dimensional system such an attachment mimics the flux attachment scenario arising out of the Chern-Simons gauge theory coupled to matter and thus lead to formation of anyons. This indicates a posssible deeper connection of these systems to fractional quantum Hall systems. This also motivates us for a further investigation into the question whether the dual theory can lead us to Hall effect for fermions in presence of a large number of vortices or vortex lattice.

In our work we assume presence of background superconductivity and formation of vortices in it, so it will be useful to discuss superconductivity as well as the topological vortex solution in the effective theories of superconductivity. We shall also review the boson-vortex duality in different systems of different dimensions briefly. In particular, we shall focus on its application to superconducting systems and FQHE. In such systems topological fields theories play a central role and often appear in the dual theory. For this reason we shall try to review some properties and application of topological terms in different condensed matter systems. We have also mentioned that anyons do appear in our work via electron-vortex attachment. The context of anyons is a proper example of the concept of statistical transmutation. Statistical transmutation, in particular bosonisation is also a topic of current research interest. For this we shall review the method of bosonization by pairing fermionic operators in 1 dimension and via flux attachment in 2 dimensional systems. This short review will help us go through the physics and methods that we shall encounter latter in this thesis.

1.1 Superconductivity and appearance of vortices:

The phenomenon of superconductivity was discovered by Kamerlingh Onnes in 1911 while studying the transport properties of Mercury (Hg) at low temperature. It was found in the experiment that below 4.2 K (liquefaction temperature of Helium) the resistivity of Hg sharply drops to zero. In general in metals resistivity arises due to scattering of electrons from the vibrating ionic background, scattering from other electrons, as well as scattering from impurities. The resistivity due to scattering from vibrating lattice ions (electron-phonon scattering) varies with temperature as $\propto T^5$, while it gets a contribution from electron-electron scattering which has a temperature dependence $\propto T^3$. Also the impurity scattering gives a constant contribution. Thus the expectation based on such ideas was that as temperature goes down to zero resistivity also drops to zero in a continuous fashion. Thus a sudden drop in resistivity at non zero temperature was surprising.

Another more surprising feature of superconductivity was discovered in 1933 by W. Meissner and R. Ochsenfeld. They showed that below the transition temperature (T_C) the magnetic flux gets completely expelled from the bulk of the superconducting material and thus the material becomes completely diamagnetic: $\chi = \dfrac{\partial M}{\partial H} = -\dfrac{1}{4\pi}$. This effect is known as Meissner-Ochsenfeld effect or popularly as Meissner effect. A phenomenological theory of electromagnetic properties of superconductors was first proposed by H. London and F. London. The key point that arises from their theory is the coherence or rigidity in the superconducting state which means that the wavefunction of electron does not change even when magnetic field is present. Let us briefly explain this before proceeding towards the microscopic description.

As explained above, superconductors possess two fundamental features: 1) perfect conductivity 2) Meissner effect. Let us try to see whether the first feature lead to the second. Hence we start by assuming the material to be a perfect conductor i.e. electrons accelerate freely under the application of electric field. So we can write

$$m\ddot{\vec{r}} = -e\vec{E}. \tag{1.2}$$

Now if the density of the superconducting electrons is n_s the current density is given by: $\vec{J}_s = -n_s e\dot{\vec{r}}$. Using this, equation (1.2) becomes

$$\dot{\vec{J}}_s = \frac{n_s e^2}{m}\vec{E}. \tag{1.3}$$

Now using Maxwell's second and fourth equations subsequently we get

$$\nabla^2\left(\frac{\partial\vec{B}}{\partial t}\right) = \frac{1}{\lambda^2}\left(\frac{\partial\vec{B}}{\partial t}\right), \tag{1.4}$$

where we defined the penetration depth as $\lambda = \sqrt{\frac{mc^2}{4\pi n_s e^2}}$. If one tries to solve Eq. (1.4) in one dimension one gets the following solution

$$\frac{\partial\vec{B}}{\partial t} = \left(\frac{\partial\vec{B}}{\partial t}\right)_{x=0} e^{-x/\lambda}. \tag{1.5}$$

This indicates that the rate of change magnetic field decays exponentially and becomes zero within the bulk of the material. Thus magnetic field inside the bulk remains constant in time and not zero. Thus this equation does not give us Meissner effect. Therefore only the assumption of perfect conductivity does not lead to Meissner effect. Thus to explain the Meissner effect one need to eliminate the time derivative arbitrarily from the Eq. (1.4) and arrive at the phenomenological equation proposed by London brothers

$$\nabla^2\vec{B} = \frac{1}{\lambda^2}\vec{B}. \tag{1.6}$$

Now using Ampere's law with the above equation one can reach to the famous London equation

$$\vec{J} = -\frac{n_s e^2}{mc}\vec{A}. \tag{1.7}$$

For such a solution of Eq. (1.6) one need to take Coulomb gauge $\vec{\nabla}\cdot\vec{A} = 0$. However the London equation can be justified using quantum mechanical arguments as proposed by F. London. In the ground state of the system the quantum expectation value of momentum of the system becomes zero i.e.

$$\langle\psi|\,\vec{p}\,|\psi\rangle = 0. \tag{1.8}$$

Now if the wave function is assumed to be rigid enough that it does not change even under application of the applied magnetic field then we have

$$\langle\psi|\left(\vec{p}-\frac{e}{c}\vec{A}\right)|\psi\rangle = 0. \tag{1.9}$$

This suggests that $\langle\psi|\,\vec{p}\,|\psi\rangle = e\vec{A}$. So if the current is defined as $\vec{J} = -\frac{n_s e}{m}\langle\psi|\,\vec{p}\,|\psi\rangle$ then the above equation leads us to

$$\vec{J} = -\frac{n_s e^2}{mc}\vec{A}, \tag{1.10}$$

which we recognize as the London equation. Thus the London equation and Hence the Meissner effect follows from the rigidity of the ground state wave function as stated above. Apart from the two fundamental properties of superconductors described above, many other important characteristics of such systems emerge out of other experiments. The electronic specific heat measurement shows a exponential decay near $T = 0K$ suggesting a gap in the energy spectrum of single particle excitations. Another important property that was established by experiments was the dependence of the transition temperature on mass of the isotope of the metal. This suggests that electron-phonon interaction is responsible for the phenomenon of superconductivity, as was also suggested independently by Fröhlich [34]. One now need a microscopic theory that captures all these phenomenological features in one frame work. Many famous physicists including Feynman, Einstein, Bohr, Heisenberg, Born attempted but could not figure out the physical picture.

It was Fröhlich's theory, based on perturbation theoretic approach, that first gives correct isotopic mass dependence for the critical field H_0 at $T = 0K$. However it predicts no separate phase with the above mentioned characteristics of superconductors. Many subsequent attempts by Bardeen, Pines, Nakajima etc could not give the correct theory untill the work of Bardeen-Cooper-Schrieffer [35] came in 1957. What they showed is that the electron-phonon interaction Hamiltonian gives rise to an attractive part when the energy difference of the two electronic states, involved in the scattering, is $\Delta\epsilon < \hbar\omega$, ω being frequency of phonons exchanged in this process. This part of the electron phonon interaction leads to a phenomenon known as Cooper instability which leads to a new ground state below Fermi surface. Let us try to briefly state the idea of Cooper instability before moving to the microscopic BCS theory.

The essential physics of BCS theory can be discussed with ina much simpler quantum mechanical prolem involving two electrons interacting via a potential V$(\vec{r}_1 - \vec{r}_2)$. One can write the Schrödinger equation of this problem in terms of relative displacement $\vec{r} = \vec{r}_1 - \vec{r}_2$, the centre of mass coordinate $\vec{R} = \frac{1}{2}(\vec{r}_1 + \vec{r}_2)$, and reduced mass of electrons $\mu = m/2$ as

$$\left[-\frac{\hbar^2}{2\mu}\nabla_r^2 + V(\vec{r})\right]\psi(\vec{r}) = \tilde{E}\psi(\vec{r}), \tag{1.11}$$

where we have eliminated the centre of mass coordinate dependence of ψ by writing $\psi(\vec{r},\vec{R}) = \psi(\vec{r})e^{i\vec{K}\cdot\vec{R}}$ and have written $E - \frac{\hbar^2K^2}{4m} = \tilde{E}$, $\vec{K}$ being the momentum of the centre of mass of the system. Here we shall consider the case in which $E = \tilde{E}$, or $\vec{K} = 0$ i.e. the electrons have equal and opposite momentum. We shall now write down the momentum space version of the above Schrödinger equation by using the Fourier transform of the wave function of the two electron system $\psi(\vec{r})$ as

$$\psi(\vec{r}) = \int \frac{d^3k}{(2\pi)^3}\psi(\vec{k})e^{i\vec{k}\cdot\vec{r}}. \tag{1.12}$$

Thus we get the following equation

$$\Delta(k) = -\int \frac{d^3k'}{(2\pi)^3}\frac{V(\vec{k}-\vec{k}')}{2\epsilon_{k'}-E}\Delta(\vec{k}'), \tag{1.13}$$

where ϵ_k is the momentum of a individual Bloch state. Also we have redefined the wave function of the system of two electron as: $\Delta(\vec{k}) = (E-2\epsilon_k)\psi(\vec{k})$. As mentioned above, the crucial point of BCS theory was that the potential due to electron-phonon interaction is attractive only for a thin shell in momentum space above fermi sphere i.e. $V(\vec{k}-\vec{k}') = -V_0$ for $\epsilon_k - \epsilon_F < \hbar\omega_D$, where ω_D is the highest possible frequency of phonon modes, called the Debye frequency. Under such assumption we can approximate the density of states of electrons by the density of states on the fermi sphere($\rho(\epsilon_F)$) and then for a constant Δ we can get from the Eq. (1.13)

$$\Delta = V_0\rho(\epsilon_F)\Delta\int_{\epsilon_F}^{\epsilon_F+\hbar\omega_D}\frac{d\epsilon}{2\epsilon-E}. \tag{1.14}$$

We now define the bound state energy to be $E_b = 2\epsilon_F - E$ and also take the assumption that $E_b << \omega_D$. With all these we obtain as a solution of above integral equation

$$E_b = 2\omega_D e^{-\frac{2}{V_0\rho(\epsilon_F)}}. \tag{1.15}$$

Thus we see that there will be a bound state of electrons regardless of smallness of the interaction potential. Such pairs are called a Cooper pair. Thus in the presence of a small attractive interaction electrons near Fermi surface form bound pairs and their energy become lower than the energy of the individual electrons on Fermi surface. Therefore a large no of such pairs form a new ground state. The instability of fermions on Fermi surface towards forming a bound pair is called Cooper instability. The new ground state of Cooper pairs becomes superconducting. As we have already discussed one of the main properties of such a ground state is the presence of a gap in the spectrum of single particle excitation. Below we shall briefly describe how such a conclusion can be drawn using mean field approach to microscopic BCS theory. The effective Hamiltonian of electronic excitations interacting via an attractive phonon mediated potential is given by

$$H = \sum_{k\sigma}\xi_{\vec{k}}c^\dagger_{k\sigma}c_{k\sigma} + \frac{1}{N}\sum_{k,k'}V_{k,k'}c^\dagger_{k\uparrow}c^\dagger_{-k\downarrow}c_{-k'\downarrow}c_{k'\uparrow}, \tag{1.16}$$

where $c^\dagger_{k\sigma}$ creates an electron at a momentum k and spin σ. Also we have written $\xi_{\vec{k}} = \epsilon_{\vec{k}} - \mu$ in the kinetic term including chemical potential. The second term describes the four Fermi interaction arising from electron-phonon interaction. To proceed we remember that the ground state is occupied by a large number of such Cooper pairs. For such a state one can safely replace the number operator of the cooper pair $\hat{N}_C = c^\dagger_{k\uparrow}c^\dagger_{-k\downarrow}c_{-k\downarrow}c_{k\uparrow}$ by it's quantum expectation value

$$c^\dagger_{k\uparrow}c^\dagger_{-k\downarrow}c_{-k\downarrow}c_{k\uparrow} \rightarrow N_0. \tag{1.17}$$

However the same will be obtained if one replaces the creation operator by a classical number $b = c_{-k\downarrow}c_{k\uparrow} \to \sqrt{N_0}$. This is called the mean field approximation which in this case means that the probablity of having Cooper pairs in this ground state is non-zero. In the mean field approximation we shall include fluctuation by assuming the difference $c_{-k\downarrow}c_{k\uparrow} - \langle c_{-k\downarrow}c_{k\uparrow}\rangle$ to be very small. Thus the second term in the above Hamiltonian can be written as

$$\langle c^\dagger_{k\uparrow}c^\dagger_{-k\downarrow}c_{-k\downarrow}c_{k\uparrow}\rangle \simeq \langle c^\dagger_{k\uparrow}c^\dagger_{-k\downarrow}\rangle c_{-k\downarrow}c_{k\uparrow} + c^\dagger_{k\uparrow}c^\dagger_{-k\downarrow}\langle c_{-k\downarrow}c_{k\uparrow}\rangle - \langle c^\dagger_{k\uparrow}c^\dagger_{-k\downarrow}\rangle\langle c_{-k\downarrow}c_{k\uparrow}\rangle. \tag{1.18}$$

Now defining the gap function as $\Delta_k = -\frac{1}{N}\sum_{k'} V_{k,k'}\langle c_{-k'\downarrow}c_{k'\uparrow}\rangle$ we can rewrite the BCS Hamiltonian under this mean field approximation as

$$H = \sum_{k\sigma} \xi_{\vec{k}} c^\dagger_{k\sigma}c_{k\sigma} + \sum_k \left(\Delta_k c^\dagger_{k\uparrow}c^\dagger_{-k\downarrow} + \Delta^*_k c_{-k\downarrow}c_{k\uparrow}\right) + \sum_k \Delta_k\langle c^\dagger_{k\uparrow}c^\dagger_{-k\downarrow}\rangle \tag{1.19}$$

To understand the nature of the excitation above BCS ground state we shall diagonalize the above Hamiltonian by the means of Bogoliubov transformation

$$\begin{aligned} c_{k\uparrow} &= u^*_k\gamma_{k\uparrow} + v_k\gamma^\dagger_{-k\downarrow}, \\ c^\dagger_{-k\downarrow} &= u_k\gamma^\dagger_{-k\downarrow} - v^*_k\gamma_{k\uparrow}, \end{aligned} \tag{1.20}$$

where the new operators $\gamma_{k\sigma}$ ($\gamma^\dagger_{k\sigma}$) create (annihilate) new particles which are linear superposition of electronic and hole excitations of the background Fermi liquid. In the diagonalization process one puts the above transformations into the BCS Hamiltonian and equate the off-diagonal terms in the γ_k basis to zero. This leads to the following diagonalized Hamiltonian

$$\begin{aligned} H_{Diag} &= \sum_k E_k\gamma^\dagger_{k\sigma}\gamma_{k\sigma} + E_0, \\ E_0 &= \sum_k \left(\xi_{\vec{k}} - E_k + \Delta_k\langle c^\dagger_{k\uparrow}c^\dagger_{-k\downarrow}\rangle\right). \end{aligned} \tag{1.21}$$

Here E_0 is the ground state energy and $E_k(= \sqrt{\xi_k^2 + |\Delta_k|^2})$ is the energy eigen value of the quasi excitations, called Bogoliubons, with momentum k. Thus we see that even at the Fermi level where $\xi_k = 0$, the spectrum of the particles is gapped and the size of the gap is given by $|\Delta_k|$. This is the reason we call this function Δ_k the gap function. One can show, using BCS theory, that this gap function obeys the following self consistency equation called the gap equation

$$\Delta_k = -\frac{1}{N}\sum_{k'} \frac{V_{k,k'}\Delta_{k'}}{2E_{k'}}\tanh\left(\frac{E_{k'}}{2k_BT}\right). \tag{1.22}$$

Very close to transition temperature $\Delta_k \to 0$. Using this into the gap equation one can easily show that that the transition temperature is given by

$$T_c = 2\frac{e^{\gamma_E}}{\pi}\frac{\hbar\omega_D}{K_B}e^{-\frac{1}{v_0\rho_F}}, \tag{1.23}$$

and that it follows the universal rule: $\frac{\Delta_0}{K_BT_c} \simeq 1.76$. This was a result which approximately true in most of the metalic superconductors discovered at that time. We can also see that the transition temperature is proportional to Debye frequency(ω_D). Now as $\omega_D \propto \frac{1}{\sqrt{M}}$, M being the mass of background lattice ions one can see that the transition temperature predicted by BCS theory obeys the correct isotopic law indicated by experiments. BCS theory also successfully explain the behaviour of electronic specific heat and also gives rise to Meissner effect. Thus microscopic BCS description gives a complete picture for the metallic superconductors.

1.1.1 From BCS theory to Ginzburg-Landau Description:

Prior to the appearance of the microscopic BCS description for superconductors a phenomenological model was constucted by V. Ginzburg and L. Landau which captured the essential features of superconductivity. Latter L.P. Gor'kov showed how such an effective description follows from the microscopic BCS theory. In this section we shall try to describe how the Ginzburg-Landau (GL) theory of superconductivity can be derived from the microscopic theory proposed by Bardeen, Cooper and Schrieffer in path integral formalism. The microscopic BCS theory is described by the following action

$$S_{BCS} = \int_0^\beta d\tau \int d^3r \left[\bar{\psi}_\sigma \left(\partial_\tau + ie\phi + \frac{1}{2m}\left(i\vec{\nabla} + e\vec{A}\right)^2 - \mu\right)\psi_\sigma - g\bar{\psi}_\uparrow\bar{\psi}_\downarrow\psi_\downarrow\psi_\uparrow\right] \tag{1.24}$$

where $\tau = it$ is the imaginary time and the inverse termperature β is defined as the periodicity of the compact imaginary time axis. ψ_σ represent the fermionic variables with spin σ and ϕ and $\vec{A}$ are the scalar and vector potential respectively. The last term is the four fermion interaction term with the potential $V(k) \sim g$, a constant. To derive GL mean field theory from this microscopic interacting theory we need to introduce a decoupling field Δ by the Hubbard-Stratonovich decoupling scheme. This method of decoupling (or linearizing) will be used more often in this thesis and thus it is a good place to set an example for it. We shall first decouple the four-fermion interaction term into two terms containing two fermion operators and the decoupling field by introducing the following Gaussian integral.

$$\frac{1}{N}\int \mathscr{D}\Delta\mathscr{D}\bar{\Delta}\exp\left[-\int_0^\beta d\tau \int d^3r \left(\frac{1}{\sqrt{g}}\bar{\Delta} - \sqrt{g}\bar{\psi}_\uparrow\bar{\psi}_\downarrow\right)\left(\frac{1}{\sqrt{g}}\Delta - \sqrt{g}\psi_\downarrow\psi_\uparrow\right)\right] = 1, \tag{1.25}$$

N being a constant. As the Gaussian integral contributes a contant, this should not change the generating functional and therefore by introducing it into generating functional we get

$$\begin{aligned}\mathcal{Z} = &\int \mathscr{D}\bar{\psi}\mathscr{D}\psi\mathscr{D}\Delta\mathscr{D}\bar{\Delta} \\ &\exp\left[-\int_0^\beta d\tau \int d^3r \left(\bar{\psi}_\sigma\left(\partial_\tau + ie\phi + \frac{1}{2m}\left(i\vec{\nabla} + e\vec{A}\right)^2 - \mu\right)\psi_\sigma + \frac{1}{g}|\Delta|^2 - \Delta\bar{\psi}_\uparrow\bar{\psi}_\downarrow - \bar{\Delta}\psi_\downarrow\psi_\uparrow\right)\right]\end{aligned} \tag{1.26}$$

In this way dealing with the interaction term becomes more easy. The decoupling field Δ represents pairs of operators of fermionic fields with opposite spin $\bar{\psi}_\uparrow\bar{\psi}_\downarrow$ or the conjugated operator. For the convenience of our calculation, we shall first convert this decoupled theory into momentum space. In momentum space the form of the generating functional is

$$\mathcal{Z} = \int \mathscr{D}\bar{\psi}\mathscr{D}\psi\mathscr{D}\Delta\mathscr{D}\bar{\Delta}$$
$$\exp\Bigg[-\sum_{p,p'} \frac{1}{\beta L^d}\Bigg(\bar{\psi}_\uparrow(p)(-i\omega_n + \epsilon_p)\delta_{p,p'}\psi_\uparrow(p') - \bar{\psi}_\downarrow(p)(i\omega_n + \epsilon_p)\delta_{p,p'}\psi_\downarrow(p')$$
$$+ \frac{1}{g}\bar{\Delta}(p)\Delta(p')\delta_{p,p'} - \frac{1}{\sqrt{\beta L^d}}\Delta(p,p')\bar{\psi}_\uparrow(p)\bar{\psi}_\downarrow(p') - \frac{1}{\sqrt{\beta L^d}}\bar{\Delta}(p,p')\psi_\downarrow(p)\psi_\uparrow(p')\Bigg)\Bigg], \tag{1.27}$$

where we have written $p = (\vec{p}, \omega_n)$ and $\epsilon_p = \frac{p^2}{2m} - \mu$. Now to arrange the term in a more compact way we write the action in Nambu-Gorkov basis $\bar{\Psi} = \left(\bar{\psi}_\uparrow, \psi_\downarrow\right)$, $\Psi = \begin{pmatrix}\psi_\uparrow \\ \bar{\psi}_\downarrow\end{pmatrix}$ and thus get

$$\mathcal{Z} = \int \mathscr{D}\bar{\psi}\mathscr{D}\psi\mathscr{D}\Delta\mathscr{D}\bar{\Delta}\exp\Bigg[-\sum_{p,p'} \frac{1}{\beta L^d}\Bigg(\frac{1}{g}\bar{\Delta}(p)\Delta(p')\delta_{p,p'} - \bar{\Psi}(p)\mathcal{G}^{-1}_{p,p'}\Psi(p')\Bigg)\Bigg], \tag{1.28}$$

where we have written

$$\mathcal{G}^{-1}_{p,p'} = \frac{1}{\beta L^d}\begin{pmatrix} (i\omega_n - \epsilon_p)\delta_{p,p'} & \frac{1}{\sqrt{\beta L^d}}\Delta(p,p') \\ \frac{1}{\sqrt{\beta L^d}}\Delta(p,p') & (i\omega_n + \epsilon_p)\delta_{p,p'} \end{pmatrix} \tag{1.29}$$

To determine the effective action in terms of the decoupling field or the mean field we now need to integrate out the fermionic field from the above generating functional. We write the relevant part of the generating functional and evaluate it as:

$$\int \prod_p d\bar{\psi}_p d\psi_p \exp\Bigg[-\sum_{p,p'} \frac{1}{\beta L^d}\Bigg(\bar{\Psi}(p)\mathcal{G}^{-1}_{p,p'}\Psi(p')\Bigg)\Bigg] = \det\left(\mathcal{G}^{-1}_{p,p'}\right) = e^{\mathrm{Tr}\ln\left(\mathcal{G}^{-1}_{p,p'}\right)} \tag{1.30}$$

Thus the effective action that we get in this procedure is

$$S_{Eff} = \frac{1}{\beta L^d}\left(\sum_{p,p'}\frac{1}{g}\bar{\Delta}(p)\Delta(p')\delta_{p,p'}\right) - \mathrm{Tr}\ln\left(\mathcal{G}^{-1}_{p,p'}\right). \tag{1.31}$$

One can try to determine the equation of motion of the decoupling field Δ from the above theory as

$$\frac{\delta S_{Eff}}{\delta \Delta} = 0. \tag{1.32}$$

It can be verified that the above variation of the action will produce the equation below for $\Delta = \Delta_0$, a constant

$$\frac{1}{g}\frac{\bar{\Delta}_0}{\beta L^3} = \frac{1}{(\beta L^3)^2}\sum_{\vec{p},\omega_n}\frac{\bar{\Delta}_0}{\omega_n^2 + \lambda_p^2}, \quad \lambda_p = \sqrt{\varepsilon_p^2 + |\Delta_0|^2}. \tag{1.33}$$

one can recognize this equation of motion of the effective theory as the famous gap equation of BCS mean field theory(with Δ being the gap), once we perform the sum over

Matsubara frequency ω_n. This equation thus shows that the effective theory carries the essential features of BCS mean field theory. One can also show with the help of this equation that close to the transition temperature T_c, the mean field or the gap function takes the form $\Delta \propto \sqrt{T_c(T_c - T)}$. So close to the transition temperature $T \to T_c$ the mean field becomes small and thus one may expand the the effective action in a perturbation series in powers of Δ. As we shall see, this series will give us the Ginzburg-Landau mean field theory.

To proceed towards this goal we start from Eq. (1.31). We now break the inverse propagator as $\mathcal{G}^{-1} = \mathcal{G}_0^{-1} + \hat{\Delta}$, where $\mathcal{G}_0^{-1}$ is the diagonal part of $\mathcal{G}^{-1}$, and $\hat{\Delta}$ is the off diagonal part containing Δ, $\bar{\Delta}$.

$$\mathcal{G}_0^{-1} = \begin{pmatrix} (i\omega_n - \epsilon_p)\delta_{p,p'} & 0 \\ 0 & (i\omega_n + \epsilon_p)\delta_{p,p'} \end{pmatrix}, \quad \hat{\Delta} = \begin{pmatrix} 0 & \frac{1}{\sqrt{\beta L^3}}\Delta(p,p') \\ \frac{1}{\sqrt{\beta L^3}}\Delta(p,p') & 0 \end{pmatrix}. \tag{1.34}$$

Then, due to smallness of Δ one can do the following expansion of the second term in the effective action

$$\mathrm{Tr}\ln\left(\mathcal{G}^{-1}_{p,p'}\right) = \mathrm{Tr}\ln\left(\mathcal{G}_0^{-1}\left(1 + \mathcal{G}_0\hat{\Delta}\right)\right) = \mathrm{Tr}\left[\ln\mathcal{G}_0^{-1} + \ln\left(1 + \mathcal{G}_0\hat{\Delta}\right)\right] \tag{1.35}$$

Now we break the term $\ln\left(1 + \mathcal{G}_0\hat{\Delta}\right)$ into the logerithmic series of matrices :

$$\ln\left(1 + \mathcal{G}_0\hat{\Delta}\right) = \sum_{n=0}^{\infty}(-1)^n\frac{\left(\mathcal{G}_0\hat{\Delta}\right)^{n+1}}{n+1} \tag{1.36}$$

Using this expansion we can write the action as

$$S_{Eff} = \frac{1}{\beta L^d}\left(\sum_{p,p'}\frac{1}{g}\bar{\Delta}(p)\Delta(p')\delta_{p,p'}\right) - \mathrm{Tr}\ln\mathcal{G}_0^{-1} - \sum_{n=0}^{\infty}(-1)^n\frac{\left(\mathcal{G}_0\hat{\Delta}\right)^{n+1}}{n+1} \tag{1.37}$$

As the matrix $\hat{\Delta}$ is an off-diagonal matrix, n = even terms i.e. where $\left(\mathcal{G}_0\hat{\Delta}\right)$ have odd powers, will have zero trace. So the expansion contains only terms with even powers of $\left(\mathcal{G}_0\hat{\Delta}\right)$. Therefore the effective action will be

$$S_{Eff} = \frac{1}{\beta L^d}\left(\sum_{p,p'}\frac{1}{g}\bar{\Delta}(p)\Delta(p')\delta_{p,p'}\right) - \mathrm{Tr}\ln\mathcal{G}_0^{-1} - \sum_{n=0}^{\infty}\frac{\mathrm{Tr}\left(\mathcal{G}_0\hat{\Delta}\right)^{2n}}{2n} \tag{1.38}$$

Let us now consider the second term in the series

$$\frac{\mathrm{Tr}\left(\mathcal{G}_0\hat{\Delta}\right)^2}{2} \tag{1.39}$$

Here the trace, represented by "Tr" means the trace in momentum as well as the trace of the matrices, where as we shall denote "tr" as the Dirac trace (or the matrix trace). Thus one may write the above term as

$$\frac{1}{2}\mathrm{Tr}\left(\mathcal{G}_0\hat{\Delta}\right)^2 = \frac{1}{2}\sum_p \langle p|\,\mathrm{tr}\left(\mathcal{G}_0\hat{\Delta}\mathcal{G}_0\hat{\Delta}\right)|p\rangle \tag{1.40}$$

To evaluate the trace further we introduce complete set of basis states in momentum space $\sum_{p'} |p'\rangle \langle p'|$ between the operators and that would result in the following expression

$$\frac{1}{2}\sum_{p,\,q} \mathrm{tr}\left(\mathcal{G}_{0\,p}\hat{\Delta}_{p,\,q}\mathcal{G}_{0\,q}\hat{\Delta}_{q,\,p}\right) \tag{1.41}$$

We also note that $\mathcal{G}_0$ is diagonal in both momentum space as well as in Nambu-Gor'kov spinor basis and due to that the above expression would become

$$\begin{aligned}
&\frac{1}{2}\sum_{p,\,q} \mathrm{tr}\left(\mathcal{G}_{0\,p}\hat{\Delta}_{p,\,q}\mathcal{G}_{0\,q}\hat{\Delta}_{q,\,p}\right)\\
&= \frac{1}{2\beta L^3}\sum_{p,\,q} \mathrm{tr}\begin{pmatrix} \mathcal{G}^P_{0\,p}\Delta(p+q)\mathcal{G}^H_{0\,q}\bar{\Delta}(p+q) & 0 \\ 0 & \mathcal{G}^H_{0\,p}\bar{\Delta}(p+q)\mathcal{G}^P_{0\,q}\Delta(p+q)\end{pmatrix}\\
&= \frac{1}{2}\sum_{p,\,q}\left[\mathcal{G}^P_{0\,p}\,\Delta(p+q)\,\mathcal{G}^H_{0\,q}\,\bar{\Delta}(p+q) + \mathcal{G}^H_{0\,p}\,\bar{\Delta}(p+q)\,\mathcal{G}^P_{0\,q}\,\Delta(p+q)\right],
\end{aligned} \tag{1.42}$$

where we use the notation $\mathcal{G}^P_{0\,p}$ represent the Green's function for the particles and $\mathcal{G}^H_{0\,q}$ represent the Green's function for the holes. One can exchange the labels p, q for the second term in the last equation and find the two terms to be identical. Also one can obtain the Green's function of holes from that of particles by time reversal operation: $\mathcal{G}^H_{0\,q} \to -\mathcal{G}^P_{0\,-p}$. Applying these we get

$$\begin{aligned}
&\frac{1}{2}\sum_{p,\,q}\left[\mathcal{G}^P_{0\,p}\,\Delta(p+q)\,\mathcal{G}^H_{0\,q}\,\bar{\Delta}(p+q) + \mathcal{G}^H_{0\,p}\,\bar{\Delta}(p+q)\,\mathcal{G}^P_{0\,q}\,\Delta(p+q)\right]\\
&= -\sum_{p,\,q}\left[\mathcal{G}^P_{0\,p}\,\Delta(p+q)\,\mathcal{G}^P_{0\,-q}\,\bar{\Delta}(p+q)\right].
\end{aligned} \tag{1.43}$$

Now with a change of variable $k = p + q$ we finally have

$$\frac{1}{2}\,\mathrm{Tr}\left(\mathcal{G}_0\hat{\Delta}\right)^2 = -\sum_{p,\,k}\left[\mathcal{G}^P_{0\,p}\,\mathcal{G}^P_{0\,p-k}\Delta(k)\,\bar{\Delta}(k)\right]. \tag{1.44}$$

Therefore, upto the square order the effective action will become

$$S_{Eff} = \frac{1}{\beta L^3}\sum_k\left[\frac{1}{g} - \frac{1}{\beta L^3}\sum_p \mathcal{G}^P_{0\,p}\,\mathcal{G}^P_{0\,p-k}\right]\Delta(k)\,\bar{\Delta}(k). \tag{1.45}$$

For $\Delta(k) = \Delta_0$, a constant, over momentum space,one can put the coefficient of $\Delta_0\bar{\Delta}_0$ i.e. $\sum_k\left[\frac{1}{g} - \frac{1}{\beta L^3}\sum_p \mathcal{G}^P_{0\,p}\,\mathcal{G}^P_{0\,p-k}\right]$ to zero. i.e.

$$\sum_k\left[\frac{1}{g} - \frac{1}{\beta L^3}\sum_p \mathcal{G}^P_{0\,p}\,\mathcal{G}^P_{0\,p-k}\right] = 0. \tag{1.46}$$

This equation gives us the expression of transition temperature which is the same as that of BCS mean field theory. Therefore the coefficient of $|\Delta|^2$ must switch sign across the transition temperature $T = T_c$. If we denote this coefficient of $\Delta(k)\,\bar{\Delta}(k)$ term as a then one can claim that near the transition temperature

$$a = \sum_k\left[\frac{1}{g} - \frac{1}{\beta L^3}\sum_p \mathcal{G}^P_{0\,p}\,\mathcal{G}^P_{0\,p-k}\right] \propto (T - T_c). \tag{1.47}$$

Therefore, from the Eq. (1.45) one can infer that below certain temperature $T = T_c$ the action becomes negative and therefore unstable. So, to make the theory stable one needs to consider higher terms in the expansion. Thus, near the transition the action can be written as an expansion in terms of even powers of the mean field or gap function Δ as

$$S_{Eff} = \frac{1}{\beta}\int d\tau d^3r \left[\frac{1}{2}a(T)|\Delta|^2 + b|\vec{\nabla}\Delta|^2 + c|\Delta|^4 + ...\right] \tag{1.48}$$

The term $|\vec{\nabla}\Delta|^2$ appears as soon as we consider the mean field or gap function Δ as function of momentum k. This is the Landau-Ginzberg mean field theory of superconductors. Thus we see that starting from the microscopic BCS theory of superconductivity and considering loop correction due to non-relativistic elctrons we land up on the BCS mean field theory in terms of mean field Δ, which represent the Cooper pair wave function or the order parameter of the theory. In general this Cooper pair wave function can depend on $\vec{k}$ in different interesting ways according to different possible angular moementum and spin states of the constituent electrons. In the simplest possible case when the Cooper pairs have zero angular momentum and zero spin, Cooper pair wave function Δ is described by a single scalar function which depends only on $|\vec{k}|$ and therefore is isotropic in momentum space and the pairing is called a "s-wave" pairing.

In this simplest case one can generalize the above Ginzburg-Landau mean field description to the Higgs model with global $U(1)$ symmetry. This is called Abelian Higgs model (AHM) and can be expressed by the following lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D^{\mu}\phi^{\dagger}D_{\mu}\phi + V(\phi^{\dagger}\phi), \quad D_{\mu}\phi = (\partial_{\mu} + ieA_{\mu})\,\phi \tag{1.49}$$

Here ϕ is the complex scalar field representing Cooper pairs (or Δ in mean field version) and being charged it couples minimally to electromagnetic gauge field A_{μ}. The potential contains terms similar to the quadratic and quartic terms in Ginzburg-Landau theory. Although this model is a toy model of superconductors near transition temperature, it has huge applications in qualitative understanding of different concepts and phenomenas both in condensed matter and high energy contexts. In particular the physics of vortices can be understood using this model and we are going to discuss this in the next section. We shall use an extended version of this model in our work and therefore discussion of this connection will be useful in understanding our work better.

1.1.2 Nielsen-Olesen Vortex solution:

We have seen in the above discussion that an effective description in terms of the order parameter or gap function emerges out of the microscopic BCS theory. The equation of motion of such an effective theory may give rise to solutions which are localized in space i.e. their total energy is contained almost in a narrow region in space, the solution would decay down to zero above a characteristic scale. Such a solution of classical field theories are called "Solitons". Here in our case we seek a similar solution which are called vortices. In ordinary terms, vorticity of a fluid means the curl of the velocity vector of the fluid i.e. it characterizes the rotational motion of a fluid. In this case the charged superfluid possess a rotational motion and the vorticity would be quantized in units of a vorticity quantum 2π. The integer multiple of 2π is of topological origin and is called a winding number. Below we shall discuss the vortex solution and its topological aspects in the context of Abelian Higgs model introduced in the previous discussions. So we start from the Lagrangian of the Abelian Higgs model

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D^{\mu}\phi^{\dagger}D_{\mu}\phi + V(\phi^{\dagger}\phi). \tag{1.50}$$

Following the work by H.B. Nielsen and P. Olesen we take the form of the potential to be $V(\phi) = m^2|\phi|^2 - \frac{\lambda}{4}|\phi|^4$. With this form of the potential one can derive the equation of motion of the complex scalar field ϕ and electromagnetic gauge field which are of the form

$$\begin{aligned} &(\partial_\mu + ieA_\mu)^2\phi = -m^2\phi + \lambda\phi^*\phi^2\,, \\ &\partial_\nu F^{\mu\nu} = ie\left(\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^*\right) + e^2A_\mu\phi^*\phi. \end{aligned} \tag{1.51}$$

To begin with we assume that the the scalar field has gone through a spontaneous symmetry breaking and its amplitude part $|\phi|$ is fixed at vacuum expectation value (vev) of the scalar field. Now to get a solution corresponding to vortex configuration we impose the following boundary condition. We assume that at spatial infinity the fields take the following form

$$\phi = \phi_0 e^{in\theta}, \quad \phi_0 = \langle 0|\,|\phi|\,|0\rangle\,. \tag{1.52}$$

This means that far from the core the field ϕ reaches to its vev, while the phase $\chi\ (= n\theta)$ is mapped to a circle at spatial infinity with winding n. This map is responsible for this solution being topological because this a mapping of the circular vacuum manifold $|\phi|^2 = \phi_0^2$ to a circle in spatial infinity. This mapping belongs to first fundamental homotopy group with $\pi_1(s_1) = \mathbb{Z}$, which is additive group of integers. One can now calculate the energy of the configuration with the above mentioned assymptotic behaviour and show that it diverges.

To make the solutions regular one need to assume that the kinetic term of the scalar field will reduce asymptotically to zero at spatial infinity. Thus we have

$$(\partial_\mu + ieA_\mu)\phi = 0. \tag{1.53}$$

Putting the asymptotic condition for the scalar field we get from the above equation: $A_\mu = -\frac{n}{e}\partial_\mu\theta$. This condition is particularly important as it leads to flux quantization. It can be readily shown by calculating the flux contained with in a loop at spatial infinity

$$\Phi = \oint A_\mu dx^\mu = -\frac{n}{e}\oint \partial_\mu\theta dx^\mu = -\frac{2\pi n}{e}. \tag{1.54}$$

This is the topological charge of vortices and as we can see it is quantized in the units of $-\frac{2\pi}{e}$. With the help of these asymptotic behaviour one can determine approximate solutions of the equations of motions. For that we first take the static case with the gauge choice $A_0 = 0$. We assume cylindrical symmetry with the axis along the Z axis. Under such assumption one may take the form of the vector potential as follows

$$\vec{A}(r) = \frac{\vec{r}\times\hat{z}}{r}A(r), \tag{1.55}$$

where r is the radial coordinate in a cylindrical coordinate system. We now notice that the flux through a region of radius r is $\Phi = 2\pi rA(r)$. One can write down the magnetic field as

$$B = \frac{1}{r}\frac{d}{dr}(rA(r)) = \frac{1}{r}\frac{d\Phi}{dr}. \tag{1.56}$$

In cylindrically symmetric case the equation of motions can be rewritten as

$$\begin{aligned}
&-\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}|\phi|\right)+\left[\left(\frac{1}{r}-eA(r)\right)^2-m^2+\lambda|\phi|^2\right]|\phi|=0,\\
&-\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(rA(r))\right)+|\phi|^2\left(e^2A-\frac{e}{r}\right)=0.
\end{aligned}\tag{1.57}$$

To obtain an approximate solution we put $|\phi|\simeq\phi_0$ in the second equation of Eq. (1.57) which is valid for the asymptotic conditions. One can obtain an exact solution under such assumptions given by

$$A(r)=\frac{1}{er}+\frac{c}{e^2}K_1(e|\phi|r)\rightarrow_{r\to\infty}\frac{1}{er}\sqrt{\frac{\pi}{2e|\phi|r}}e^{-e|\phi|r}.\tag{1.58}$$

Thus using the above mentioned expression of magnetic field B we obtain

$$B\rightarrow_{r\to\infty}\frac{c}{e}\sqrt{\frac{\pi|\phi|}{2er}}e^{-e|\phi|r}\tag{1.59}$$

So we see that there is a characteristic length scale $\Lambda=\frac{1}{e|\phi|}$. This length scale measures the region over which the magnetic field differ significantly from zero. One can determine the value of $|\phi|^2$ in the symmetry broken phase to be $\phi_0=\frac{m^2}{\lambda}$. Now one can ask the question that how the scalar field varies inside the normal region of a vortex. To determine that we assume $\phi=\phi_0+\rho$, ρ being the fluctuation from the vacuum value. One can now put this ansatz into the first equation of Eq. (1.57) and determine that

$$\rho\sim e^{-\sqrt{\lambda}\phi_0 r}.\tag{1.60}$$

Thus there is another characteristic scale $\xi=\frac{1}{\sqrt{\lambda}\phi_0}$ beyond which the scalar reaches to its vev. Thus we see that the solution with these two characteristic length scales define a tube like object in 3+1 dimensional theory. The two length scales give an internal structure to the vortices. But in the special case where $\xi\sim\Lambda$ the solution becomes a flux tube with the core width $\xi\sim\Lambda$. In the London limit where $\lambda\to\infty$ the width of the tube reduces to zero and the solution becomes a string with no internal structure.

Vortices are of special interest to us as in our work we shall be mostly concerned in a phase of superconductors where appearance of vortices is allowed. In a type II superconductor there is a phase called mixed phase in which vortices are seen to form. Vortices and their dynamics are very important in phenomenological aspects of different superconductors as they can give rise to different phases of the system like Abrikoshov vortex lattice phase, vortex liquid phase, etc. The concept of topological vortices are also useful in case of spin systems like 2 dimensional XY model, in the definition of composite fermions in quantum Hall systems, in the definition of anyons etc. We shall see in the next section that using the idea of multivaluedness of the phase of the scalar field in presence of vortex one may redefine the action of the scalar (or spin systems) such that the new action expresses vortices and their interaction explicitly. Such transformation of the original action (or Hamiltonian) is called dualization.

1.2 Dualities in Field Theory:

Dualities provide a powerful tool to understand phenomena which are not tractable by perturbation techniques. Dualities usually relate strongly coupled sector in one theory to the weakly coupled sector in another theory, in particle physics, string theory, statistical physics and also condensed matter physics. One class of duality transformations in quantum field theory involves exchanging a differential p-form A_p in D-dimensions with a $D-p-2$-form A_{D-p-2} by the Hodge duality of their exterior derivatives, $\mathrm{d}A_p = {}^*\mathrm{d}A_{D-p-2}$. For free fields in topologically trivial spacetimes, both are equivalent descriptions, but interactions or topological obstructions can break this duality. Dualities in interacting theories, when they can be constructed, can lead to deep mathematical and physical insights. In two and three dimensions, (anti)self-dual configurations of gauge fields interacting with scalars correspond to solitons. In four dimensions, where the dual of a 1-form is also a 1-form, the (anti)self-dual configurations of Yang-Mills gauge fields minimize the action of instantons. A particularly interesting duality in four dimensions is that between a scalar and a two-form, which persists when there is a gauge field coupled to the scalar. The scalar field is now compact and the dual two-form appears in a topological $B \wedge F$ interaction with the gauge field. If there is no other field in the theory, this is a duality between the strongly coupled Abelian Higgs model(AHM) and topological mass generation mechanism in four dimensions.

1.2.1 Duality in lattice models:

Similar dualities also exist in various spin models of condensed matter and statistical physics. Let us start our discussion from the X-Y duality or charge-vortex duality and show how it can be extended to AHM from which our work will start. We shall illustrate some of its application in the context of condensed matter as well.

There exists a duality which maps the partition function of a disordered lattice bosons at $T = 0$ to another model which is isomorphic to a anisotropic lattice superconductor with vortex like excitations. This duality is called boson-vortex duality and applicable to a number of possible ground states of a boson system. Example of such duality are correspondence between a density wave boson insulator and Abrikoshov flux lattice phase, correspondence between superfluid phase of the boson system and non-superconducting phase of flux line liquid etc. Here we shall review the correspondence between X-Y model in 2d and 3d and coulomb gas of vortices in respective dimension.

X-Y Duality in 2d:

Let us consider a two dimensional square lattice and on each lattice point there is a spin $S_n = e^{i\theta_n}$, $-\pi < \theta \leq \pi$. The spins interact through a nearest neighbour coupling as expressed by the following partition function [36]

$$\mathcal{Z} = \int \prod_n d\theta_n \exp\left(-\beta \sum_{\langle r,r' \rangle} S_r \cdot S_{r'}\right) = \int \prod_n d\theta_n \exp\left(-\beta \sum_{n\mu} \cos\left(\theta_{n+\hat{\mu}} - \theta_n\right)\right). \quad (1.61)$$

To derive the dual theory we first write down Eq. (1.61) using Fourier expansion

$$\exp\left(-\beta \cos\left(\Delta_\mu \theta_n\right)\right) = \sum_{k_{\mu,n}=-\infty}^{\infty} I_{k_{\mu,n}}(\beta) e^{ik_{\mu,n}\Delta_\mu \theta_n}, \quad (1.62)$$

where $I_{k_{\mu,n}}$ are modified Bessel functions. Also we have introduced a new variable $k_{\mu,n}$ which can vary over all set of integers on each link. Putting this into the expression of Eq. (1.61) we get

$$\mathcal{Z} = \prod_{n\mu} \int D\theta e^{(-\beta\cos(\Delta_\mu \theta_n))} = \sum_{k_{\mu,j}=-\infty}^{\infty} \prod_{n\mu} I_{k_{\mu,j}}(\beta) \int D\theta e^{ik_{\mu,j}\Delta_\mu \theta_j}. \tag{1.63}$$

As a next step we can shift the lattice derivative Δ_μ from θ_n to the new variables $k_{\mu,n}$ by using sum by parts. If now one performs the integrals over θ_ns we shall get

$$\mathcal{Z} = \prod_{n\mu} \int D\theta e^{(-\beta\cos(\Delta_\mu \theta_n))} = \sum_{k_{\mu,n}=-\infty}^{\infty} \prod_{n\mu} I_{k_{\mu,n}}(\beta) \prod_n \delta\left(\sum_\mu \Delta_\mu k_{\mu,n}\right). \tag{1.64}$$

The delta function for each n gives rise to the constraint $\sum\limits_\mu \Delta_\mu k_{\mu,n} = 0$ at each lattice point which suggests divergencelessness of $k_{\mu,n}$ at each lattice site. This constraint can be satisfied if we write $k_{\mu,n}$ in the following way

$$k_{\mu,n} = \sum_\nu \varepsilon_{\mu\nu} \Delta_\nu \phi_n, \tag{1.65}$$

where the new integer valued variable ϕ_n can varie over all integers. Thus performing the sums over the delta function we can get the following generating functional.

$$\mathcal{Z} = \prod_{n\mu} \sum_{\phi_n=-\infty}^{\infty} \exp\left[\ln\left(I(\beta)_{\left(\sum_\nu \varepsilon_{\mu\nu}\Delta_\nu \phi_n\right)}\right)\right]. \tag{1.66}$$

Thus we can see that the initial theory of Eq. (1.61) is mapped to the theory of Eq. (1.66) which is written in terms of a new integer valued field ϕ_n at each lattice site n. Now one can analyse different phases of this dual theory. For the high temperature regime one can show that the partition function takes the following form by writing modified Bessel function in integral form

$$\prod_{n,\mu} \sum_{\phi_n=-\infty}^{\infty} \left[\left(1+\frac{\beta^2}{2}\right)\delta\left[\left(\sum_\nu \varepsilon_{\mu\nu}\Delta_\nu\phi_n\right)^2\right] + \beta\delta\left[\left(\sum_\nu \varepsilon_{\mu\nu}\Delta_\nu\phi_n\right)^2 - 1\right] + \beta^2\delta\left[\left(\sum_\nu \varepsilon_{\mu\nu}\Delta_\nu\phi_n\right)^2 - 4\right] + \ldots\right]. \tag{1.67}$$

We can see clearly from the above equation that for $\beta << 1$, when the original variables are disordered, the integer valued variables ϕ_n can take configurations where variation of ϕ is nearly zero i.e ϕ_n becomes nearly same at all lattice sites. As β is increased the system becomes more and more disordered in ϕ i.e. higher energy configurations of the integer valued parameter ϕ_n becomes more likely with $|\Delta_\mu \phi_n|$ taking larger and larger values.

To analyse low temperature limit of the theory in dual description one usually writes Eq. (1.61) in terms of Villain approximation [1] as follows. In the limit $\beta >> 1$ the only important configuration in the partition function will be those for which $\cos(\Delta_\mu \theta_n)$ is close to 1, which means $\Delta_\mu \theta_n$ is close to zero. Thus one can keep only quadratic term in expansion of $\cos(\Delta_\mu \theta_n)$. But we know for $\Delta_\mu \theta_n = 2\pi m$ (m being an integer), $\cos(\Delta_\mu \theta_n) = 1$ so for generalization we need to take contribution of all integers. For that

[1] See Appendix .1 for details.

we can introduce an auxiliary integer valued vector field $J_{n\mu}$ associated with lattice links and we write

$$\mathcal{Z} \simeq e^{2N\beta} \prod_{n\mu} \sum_{J_{n,\mu}=-\infty}^{\infty} \int D\theta \exp\left(-\frac{\beta}{2} \sum_{n\mu} \left(\Delta_\mu \theta_n - 2\pi J_{n\mu}\right)^2 \right). \tag{1.68}$$

This approximate description of the 2 dimensional X-Y model can now be taken to a dual picture in the following way. We proceed by linearizing the squared term in the exponent of Eq. (1.68) by intoducing a new integer valued vector field as shown below

$$\mathcal{Z} = (2\beta e^{2\beta})^N \prod_{n\mu} \sum_{J_{n,\mu}=-\infty}^{\infty} \int D\theta \int Dk_\mu \exp\left(\sum_{n\mu} -\frac{1}{2\beta} k_{\mu,n}^2 + i k_{n,\mu} \left(\Delta_\mu \theta_n - 2\pi J_{n\mu}\right) \right). \tag{1.69}$$

Again, as in the previous analysis, one can shift the lattice derivative over the new variable $k_{n,\mu}$ by sum by parts. one can then perform the integral over each θ_n to get the following generating functional

$$\mathcal{Z} = (4\pi\beta e^{2\beta})^N \prod_{n\mu} \sum_{J_{n,\mu}=-\infty}^{\infty} \prod_n \delta\left(\sum_\mu \Delta_\mu k_{n,\mu} \right) \int Dk_\mu \exp\left(\sum_{n\mu} -\frac{1}{2\beta} k_{\mu,n}^2 - 2\pi i k_{n,\mu} J_{n\mu} \right). \tag{1.70}$$

We can now satisfy this constraint by writing $k_{n,\mu} = \sum_\nu \varepsilon_{\mu\nu} \Delta_\nu \phi_n$ in terms of a new integer valued scalar field ϕ_n. Integrating over the delta functions with the above mentioned resolution of δ-function constraint we get the following theory

$$\mathcal{Z} = (4\pi\beta e^{2\beta})^N \prod_{n\mu} \sum_{J_{n\mu}} \int D\phi \exp\left(\sum_{n\mu} \left(-\frac{1}{2\beta} (\Delta_\mu \phi_n)^2 + i2\pi \phi_n m_n \right) \right), \tag{1.71}$$

where $m_n = \sum_\nu \epsilon_{\mu\nu} \Delta_\nu J_{n\nu}$ is a new integer valued field defined on each lattice site. These variables can be interpreted as the vorticity of the angles θ_n in the original X-Y model. So the model expressed by Eq. (1.61) can be described in terms of the topological excitations m_n and a spin wave expressed by the continuous variable ϕ_n. We can in principle integrate out ϕ_n from the partition function and will be left with a Coulomb gas of vortices interacting via a Coulomb like potential in 2d which is of $\log r$ type. Due to this potential vortices can form bound pairs at low temperature and spin-spin correlation is zero. But there exists a phase transition at some $T = T_c$ such that at $T > T_c$ the free energy becomes minimum when isolated vortices are separated by arbitrarily large distance. Thus a plasma of vortices form and spin-spin correlation becomes screened($\sim e^{-\mu r}$). This class of phase transitions are famously known as BKT transition after the names of the scientists Berezinskii, Kosterlitz, Thouless.

3d Boson-Vortex duality:

There exists a similar duality in 3d which maps a 3d X-Y model and a Coulomb gas of vortex like excitations [37] which can further be mapped to a theory of superconductors.

The 3d X-Y model is exactly similar to that in 2d and is given by

$$\mathcal{Z} = \int_\theta \exp\left[\beta \sum_{n,\mu} \cos\left(\theta_{n+\hat{\mu}} - \theta_n \right) \right]. \tag{1.72}$$

Here the index μ contains three dimensions and hence it is a 3d X-Y model. This, under Villain approximation ($\beta >> 1$), can be expressed as the following partition function

$$\begin{aligned}\mathcal{Z} &= \exp\Big[-F(T, e^2 = 0)\Big] \\ &= \prod_n \int \frac{d\theta_n}{2\pi} \prod_{n\mu} \sum_{m_{n\nu}=-\infty}^{+\infty} \exp\left[-\frac{1}{2T}\sum_{n\mu}(\theta_{n+\hat{\mu}} - \theta_n - 2\pi m_{n\mu})^2\right].\end{aligned} \tag{1.73}$$

Through similar duality transformation shown in previous discussion one can map the above approximate partition function into the following

$$\mathcal{Z} = \int \prod_{n\mu} a_{n\mu} \sum_{m_{n\mu}} \exp\left[-\frac{T}{4}\sum_{n\nu\sigma} f_{n\nu\sigma}^2 + 2\pi i \sum_{n\sigma} a_{n\sigma} M_{n\sigma}\right], \tag{1.74}$$

where we have defined

$$f_{n\nu\sigma} = (a_{n+\hat{\nu}\sigma} - a_{n\sigma}) - (a_{n+\hat{\sigma}\nu} - a_{n\nu}). \tag{1.75}$$

Here $a_{n\mu}$ is a new vector field which emerge in the process of dualization and $f_{n\nu\sigma}$ is the corresponding field strength as can be understood from Eq. (1.75). $M_{n\sigma}$ is an integer valued vector field defined as a curl of $m_{n\mu}$. It is obvious from Eq. (1.74) that these integer valued variables $M_{n\sigma}$ interact via a Coulomb potential. The physical interpretation of these variables can be extracted by calculating the current in presence of external magnetic field. This can be done using Eq. (1.73) by coupling it to a magnetic vector potential through the replacement $(\theta_{n+\hat{\mu}} - \theta_n) \to (\theta_{n+\hat{\mu}} - \theta_n - B_{n\mu})$ and then applying the definition

$$\langle J_{n\mu}\rangle = -\frac{\delta}{\delta B_{n\mu}}\left(-logZ[B]\right) = \frac{1}{T}\left(\theta_{n+\hat{\mu}} - \theta_n - m_{n\mu}\right), \tag{1.76}$$

where $B_{n,\mu}$ is the magnetic field at a lattice cite n. Thus we can see the curl of the above current is proportional to $M_{n\mu}$ and hence M represents vorticity in the field variable θ_n of the original 3 dimensional X-Y model. Finally one can take the Eq. (1.74) further into the following one

$$\mathcal{Z} = \lim_{t\to 0}\int \prod_{n\mu} da_{n\mu}\frac{d\theta_n}{2\pi}\sum_{m_{n\sigma}} \exp\left(-\frac{1}{4\beta(2\pi)^2}\sum_{n\nu\sigma} F_{n\nu\sigma}^2 - \frac{1}{2t}\sum_{n\sigma}\left[\theta_{n+\hat{\sigma}} - \theta_n - A_{n\sigma} - 2\pi m_{n\sigma}\right]^2\right). \tag{1.77}$$

This action expresses a superconductor in presence of an electro-magnetic field $A_{n\sigma} = 2\pi a_{n\sigma}$ and vortex like topological excitations expressed by $M_{n\mu}$. Thus we see from the above discussions that the boson-vortex duality maps the partition function of a 3d X-Y model to a model of bosonic vortices interacting via the field A_μ. This model, known as Abelian Higgs model, has two phases which corresponds to the two different phases of X-Y model [38]. The first phase is a superconductor in which vortices condense and the gauge field A_μ obtains a mass and excitations become gapped. This phase is dual to the insulating phase of the X-Y model. Another phase is an insulating one in which spectrum contains a gapless photon. This phase is dual to the superfluid phase of X-Y model.

1.2.2 Duality in continuum field theory:

We have given a very brief outline of boson-vortex duality in a 2 dimensional and 3 dimensional lattice X-Y models and has discussed some important physical implications of them. One can also devolop similar duality in case of different non-relativistic and relativistic[2] field theories in spatial continuum. The dual theories helps to understand phenomenon of fractional quantum hall effect(FQHE) hierarchy for 2 dimensional electron gas in high magnetic field and leads to an effective string theory starting from 3+1 dimensional Abelian Higgs model. Let us briefly describe the dual formulation in the above mentioned cases.

Non-relativistic electron gas in 2 dimension:

Let us start first from a system of gas of electrons interacting via a Coulomb interaction expressed by the following Hamiltonian

$$H = \frac{1}{2m}\int d^2x|\vec{D}\psi(x)|^2 + \frac{1}{2}\int d^2xd^2x'(\psi^\dagger(x)\psi(x)-\bar{\rho})V(x-x')(\psi^\dagger(x')\psi(x')-\bar{\rho}), \quad (1.78)$$

wherte ψ represents electrons confined in a 2d space. $\vec{D} = -i\vec{\partial} - e\vec{A}$ is the covariant derivative and $\vec{\nabla}\times\vec{A} = B$, B being the external magnetic field. $V(\vec{x}-\vec{x}')$ is the Coulomb potential between electrons and $\bar{\rho} = \langle\psi^\dagger(x)\psi(x)\rangle$ is the average density of electrons. It is well known, in the context of FQHE, that there is a phenomenon of flux attachment in which each electron attaches to k no of flux quanta or vortices. This attachment of flux leads to a statistical transmutation making the flux attached particles bosonic(in general it leads to anyons). Thus the system becomes a bosonic system and one can in principle apply dual transformation to this system. This dual Lagrangian may have potential application in explaining FQHE hierarchy. To realize flux attachment mathematically one need to define a new field variable ϕ in terms of original fermionic variables ψ as [39]

$$\phi^\dagger = \exp\left[-ik\int dz'\Theta(z-z')\psi^\dagger(z')\psi(z')\right]\psi^\dagger. \quad (1.79)$$

It can be shown that the correlation function between ϕ operator in the Laughlin states decays algebraically with distance

$$\Gamma_\phi(z-z') = \langle\Psi|\phi^\dagger(z)\phi(z)|\Psi\rangle \propto |z-z'|^{-k/2}. \quad (1.80)$$

where $|\Psi\rangle$ is the Laughlin state describing an incompressible quantum liquid at a filling factor [3] $1/k$. This ϕ field represents bosons which form out of fermions created by $\phi^\dagger$ and k attached flux quantas. Noticing the fact that the density of the original fermions and these new bosons are same ($\phi^\dagger(z)\phi(z) = \psi^\dagger(z)\psi(z)$) one can invert the relation Eq. (1.79) to obtain

$$\psi^\dagger = \exp\left[-ik\int dz'\Theta(z-z')\phi^\dagger(z')\phi(z')\right]\phi^\dagger. \quad (1.81)$$

With this relation one can covert the many body Hamiltonian in Eq. (1.78) in the following form

$$H = \frac{1}{2m}\int d^2x|(\vec{D}+\vec{a})\phi(x)|^2 + \frac{1}{2}\int d^2xd^2x'(\phi^\dagger(x)\phi(x)-\bar{\rho})V(x-x')(\phi^\dagger(x')\phi(x')-\bar{\rho}), \quad (1.82)$$

[2]See Appendics .3 for dualization in 2+1 dimensional relativistic model.

[3]The filling factor ν is defined as ratio of total the number of particles and total number of flux quantas.

where we have used the following definition of the fictitious gauge field a_i as

$$a_i = k \int d^2x' \partial_i^x \Theta(x - x')\rho(x') = k\epsilon_{ij} \int d^2x' \frac{x_j - x_j'}{|\vec{x} - \vec{x}'|^2}\rho(x'). \tag{1.83}$$

This newly introduced vector field is called the "statistical vector potential". This expression can be seen as the solutions of the following equation

$$\varepsilon_{ij}\partial_i a_j = k\rho, \quad \rho = \phi^\dagger\phi, \tag{1.84}$$

which expresses the fact that with each boson particle there are k attached fluxes. One can also obtain, using the equation of continuity and the above "flux attachment" condition, the following equation

$$\epsilon_{ij}\partial_0 a_j = kJ_i, \tag{1.85}$$

where J_i is the particle current. These two equations Eq. (1.84) and Eq. (1.85) can be obtained by plugging in a Chern-Simons term of the form $\epsilon_{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda$ for the statistical gauge field.

$$\mathcal{L} = i\bar{\phi}\left(\frac{\partial_0}{i} + a_0\right)\phi + \left|\left(\vec{D} + \vec{a}\right)\phi(x)\right|^2 + \frac{1}{2}(\bar{\phi}\phi - \bar{\rho})(x)V(x - x')(\bar{\phi}\phi - \bar{\rho})(x') - \frac{i}{k}\epsilon_{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda. \tag{1.86}$$

Now we shall apply duality transformation to this bosonic Lagrangian. Here we shall briefly out line the steps, which will be same for relativistic cases as well. First we decouple the second term by using Hubbard-Stratonovich decoupling scheme by introducing a gausssian integral over a new field called a decoupling field.

$$\int \mathcal{D}\vec{P} \exp\left[-\int d\tau dx^2 \frac{m}{2\rho}\vec{P}^2\right]. \tag{1.87}$$

The Gaussian integral contributing a constant factor will not change the partition function. This new decoupling field, when suitably transformed, linearizes the second term in the above lagrangian.

We have seen that in lattice field theories topological excitations enter the action through the periodicity of the phase of the spin variables and this is implemented through writing the action in Villain prescription. Here, to take into acount the presence of vortices in the bosonic matter, we decompose the phase of scalar into two parts- one is the single valued part (called spin wave) and another is the singular or multivalued part. Integrating out the single valued part we shall get a functional delta function imposing the constraint $\partial_\mu P_\mu = 0, P_0 = \rho$. This constraint can be satisfied by writing $P_\mu = \epsilon_{\mu\nu\lambda}\partial_\nu b_\lambda$, where b_μ is a new vector field. Now if we integrate over the auxiliary field P_μ we shall get the dual Lagrangian as

$$\begin{aligned}\mathcal{L} = ib_\mu(J_{v\mu} - e\varepsilon_{\mu\nu\lambda}\partial_\nu A_\lambda) + \frac{m}{2\rho}(\partial_i b_0)^2 + \frac{m}{4\rho}(\vec{\nabla} \times \vec{b})^2 + \frac{1}{2m}(\vec{\nabla}\rho^{1/2})^2 \\ + \frac{1}{2}(\rho - \bar{\rho})(x)V(x - x')(\rho - \bar{\rho})(x') + \frac{\kappa}{2}\epsilon_{\mu\nu\lambda}b_\mu\partial_\nu b_\lambda,\end{aligned} \tag{1.88}$$

where the $J_{v\mu}$ represents the current of vortices in the dual theory which couples to the dual gauge field b_μ. It is defined as curl of gradient of the multivalued phase of the bosonic field ϕ. The equation Eq. (1.86) is same as that proposed by Zhang, Hansson and Kivelson in their paper [6, 40] to describe FQHE. One can also show that the dual Lagrangian Eq. (1.88) contains the features of FQHE namely one can show that at fractional filling $\nu = \dfrac{1}{\kappa}$ external magnetic field is exactly cancelled by statistical magnetic filed and thus mean number of vortices vanishes. Thus FQHE happens at vortex vacuum. In such a state one can also calculate the induced current in presence of external electromagnetic field by writing the effective Lagrangian in terms of electromagnetic gauge field $A_\mu = \delta A_\mu + \langle A_\mu \rangle$ ($\langle A_\mu \rangle$ is the external electromagnetic field) and then taking a derivative with respect to $\langle A_\mu \rangle$. Following this one can obtain $\sigma_{xx} = 0, \quad \sigma_{xy} \sim \dfrac{e^2}{\kappa}$.

If the filling fraction $\nu \neq \dfrac{1}{k}$ a non zero density of vortices $\langle \rho_v \rangle = H_0 - \kappa \bar{\rho}$ is induced. Here, due to the particle-vortex duality, we treat vortices as particles. So the particle density would act as the effective magnetic field seen by the vortices. This can also be understood from the first term of the Lagrangian of Eq. (1.88) and remembering that $\rho = \epsilon_{ij} \partial_i b_j$. Thus the vortices can also form a Laughlin liquid themselves and can also bose condense when the effective density of the vortices seen by the new bose particles (vortices of the original bosonic fields) will effectively become zero. Let us assume that this happens at the filling fraction $\tilde{\nu} = \dfrac{\rho_v}{\rho} = \dfrac{1}{2p}$ for the boson particles in the dual theory. It is obvious that in this condensed phase the Hall conductivity for these dual bosons will be $\sigma_{xy} = \dfrac{e^2}{2p}$. In analogy to the original electron case one may understand that in this case composite particles may form out of a dual boson and 2p of their vortices. To implement this flux attachment to the vortices we add some additional terms as $\dfrac{i}{2} \dfrac{1}{2p} \epsilon_{\mu\nu\lambda} \tilde{a}_\mu \partial_\nu \tilde{a}_\lambda - i J_{v\mu} \tilde{a}_\mu$, $\tilde{a}_\mu$ is the new statistical gauge field which attaches to each vortex in the dual theory. When $\langle \rho_v \rangle = \dfrac{\bar{\rho}}{2p}$ one can show that the magnetic field due to statistical gauge field $\vec{\nabla} \times \vec{\tilde{a}} = 2p\rho_v$ cancells the magnetic field $\vec{\nabla} \times \vec{b} = \bar{\rho}$. Thus dual bosonic particles can condense in net zero magnetic field. The original filling fraction at which this happens is

$$\nu = \frac{1}{k + \frac{1}{2p}}, \tag{1.89}$$

as the number of flux quanta per unit charge is $\dfrac{1}{k + 1/2p}$. This is the starting point of FQHE hierarchy scheme [41] through which different fractional filling at which FQH plateaus are obtained. To make the process more explicit we write $J_{v\mu} = \epsilon_{\mu\nu\lambda} \partial_\nu \tilde{b}_\lambda$ into above lagrangian and then integrating out both $\tilde{a}$ and $\tilde{b}$ we have

$$\mathcal{L} = \frac{m}{2\bar{\rho}} |(\vec{\nabla} \times \vec{b})|^2 + \frac{1}{2}((\vec{\nabla} \times \vec{b})_0) V((\vec{\nabla} \times \vec{b})) + \left(\kappa + \frac{1}{2p}\right) \frac{i}{2} \epsilon_{\mu\nu\lambda} b_\mu \partial_\nu b_\lambda. \tag{1.90}$$

We can see that this Lagrangian of Eq. (1.90) is exactly same with Eq. (1.88) with no external electromagnetic field and κ replaced by $\kappa + \dfrac{1}{2p}$. Finally, if the filling factor does not satisfy the form $\nu = \dfrac{1}{\kappa + \frac{1}{2p}}$ then there will agin be finite density of vortices in phase of dual bosonic particles. Then we need to reiterate the same procedure to reach a Lagrangian of the same form with $\kappa + \dfrac{1}{2p}$ replaced by $\kappa + \dfrac{1}{2p + 1/2p'}$ and one can see a plateau at

$$\nu = \frac{1}{\kappa + \frac{1}{2p + \frac{1}{2p'}}}. \tag{1.91}$$

In this way all fractional filling at which FQHE plateaues are seen can be obtained.

Abelian Higgs model and effective string theory in 3+1d:

Another very interesting application of the boson-vortex duality can be shown by applying it to 3+1d relativistic Abelian Higgs model. We have already seen that in 2d the Hamiltonian describing a X-Y model is mapped to point vortex like excitation represented by the integer valued vorticity vector fields on lattice links. In the case of 3+1d theory of a charged scalar field minimally coupled to EM gauge field (AHM) can be mapped to theory of vortices with gauge interaction described in terms of a two-form field. The vortices here are extended tube like objects and the central points of the vortices form a string like structure. So the vorticity can be mapped to the world sheet of this string. There are many literature in which authors have tried to quantize this string theory [42]. Such description may be useful in describing QCD ground state [43]. Let us briefly out line the duality below

One starts from the Abelian Higgs model expressed by the following

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D^{\mu}\phi^{\dagger}D_{\mu}\phi + V(\phi^{\dagger}\phi), \tag{1.92}$$

where terms carry their usual definition. One need to follow the same prescription as in the 2 dimensional case. In the presence of a vortex like topological field configuration one can decompose the phase of the bosonic field as $\chi = \chi^r + \chi^s$, where χ_r is the single valued part expressing irrotational current of the scalar field in symmetry broken phase and χ_s is the multivalued part. As we already discussed in the definition of ANO vortices, this multivaluedness comes due to the mapping the phase χ of the scalar field to a spatial loop at infinity. Therefore in this case and for a static vortex the multivalued phase field can be expressed as a function of the the azimuthal angle ϕ like $\chi = n\phi$. Thus the origin of the coordinate system, with respect to which ϕ is defined, becomes the core of the vortex. In the process of dualization this χ_r is dualized to a new 2-form gauge field $B_{\mu\nu}$ and the electromagnetic gauge field A_μ is dualized to A^m_μ which is magnetic gauge field (naturally couples to a magnetic charge).

Now one defines $\varepsilon^{\mu\nu\rho\lambda}\partial_\mu\partial_\nu\chi^s = \Sigma^{\rho\lambda}$ as the vorticity which is the curl of gradient of the multivalued phase field. For a straight string this gives a delta function $\delta^2(\rho-\rho_0)$ expressing the location of the vortex on a plane perpendicular to the vortex axis at $\rho = \rho_0$

$$(\partial_x\partial_y - \partial_y\partial_x)\chi^s = 2n\pi\delta^2(\rho-\rho_0). \tag{1.93}$$

When this is written in a relativistic form we get

$$\Sigma^{\rho\lambda} = 2\pi n \int_\Sigma d\sigma^{\rho\lambda}(x(\xi))\delta^4(x - x(\xi)) \tag{1.94}$$

This is just the world sheet spanned by a string having coordinate $x = x(\xi, \tau)$, where ξ and τ are the parameters parametrizing the world sheet. This world sheet couples to the

2-form gauge field. The dual field theory of interacting strings thus obtained is given by

$$\mathcal{S} = \mathcal{S}_{string} + \int d^4x\bigg(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}v^2\partial_\mu f\partial^\mu f + \frac{1}{12f^2}H^{\nu\rho\lambda}H_{\nu\rho\lambda} - \frac{v}{2}B_{\rho\lambda}\Sigma^{\rho\lambda} - \frac{ev}{2}\varepsilon^{\mu\nu\rho\lambda}\partial_\nu B_{\rho\lambda}A_\mu + V(f^2)\bigg), \tag{1.95}$$

where the $\mathcal{S}_{string}$ represents the dynamics of the effective strings and is expressed as

$$\mathcal{S}_{string} = \mu\int d^2\sigma\sqrt{g} + \frac{11}{48\pi}\int d^2\sigma\,(\partial_a \ln\sqrt{g})^2 + \beta\int d^2\sigma\sqrt{g}\,(\partial_a t_{\mu\nu})^2 \tag{1.96}$$

which can be obtained by calculating the Jacobian of the transformation from the phase variable χ_s to the string coordinate $x(\xi,\tau)$ [42]. Here the first term is just like the Nambu-Goto term of microscopic strings and g represents determinant of the induced metric on world sheet: $g = det|g_{ab}|$, $\partial_a x_\mu \partial_b x_\mu$. Also $t_{\mu\nu}$ represents extrinsic curvature of the string and is given by: $t_{\mu\nu} = \frac{\epsilon_{ab}}{\sqrt{g}}\partial_a x_\mu \partial_b x_\nu$. Therefore we can see that an effective theory of strings can appear from the theory of complex scalar field by the application of duality. This poses an oportunity test predictions of such theories in low energy system like superconductors near transition temperature where the GL mean field theory is valid.

Apart from the charge vortex duality, discussed in various contexts so far, there exists other dualities devoloped in recent past. Historically first ever proposed dual relationship was by Kramers and Wannier [44] in 1941 in the field of statistical physics. They had shown that $d = 2$ Ising model at temperature T can be written as another Ising model with inverse temperature $1/T$. There exists dualities which maps a theory of fermions into a theory of bosons called boson-fermion duality(or bosonization) and those mapping a fermionic theory to another called fermion-fermion duality [45, 46]. Recently these dualities have been applied to understand the FQHE at half filled Landau level [47]. Finally we would like to mention another duality which is applied very much in certain regions of condensed matter physics - it is the gauge/gravity duality or AdS/CFT duality. Although it is discovered in the contexts of string theory it has found its relevence in the contexts of strongly correlated electron systems like Mott insulators and non-Fermi liquid behaviour observed in various systems including the pseudogap region of High T_c cuprate superconductors [33].

1.3 Topological Field Theories

With the newly started categorization of phases of matter with topological quantum numbers, topological QFTs or TQFTs have become important. Here we shall describe two topological terms (gauge theories) in 2+1d and in 3+1d that are most useful in describing behavious of different condensed matter systems. One of these are Chern-Simons term in 2+1d and another is BF term in both 2+1d and 3+1d. These are called topological terms as they are written by contracting all space time indices with Levi-Civita tensor in that manifold and has no metric dependence. Thus such terms depend only on the topology (or global properties) of the manifold and not on the local geometry(through metric). Another important property of such a system is that the Hamiltonian gets no contribution from such a term. This suggests that the phase described by such theories contain no local excitations. Such theories often summerize the response of the system under application of some external perturbation. As an example Chern-Simons term with proper coefficient can give rise to fractional off diagonal conductivity seen in FQHE [40]. Another such term in 3+1d is the axion term or the θ term given by : $\frac{\theta}{32\pi^2}\varepsilon^{\mu\nu\rho\lambda}F_{\mu\nu}F_{\rho\lambda}$ which gives the electro-magnetic response of a 3+1d topological insulator [48, 49]. We shall not discuss about this term in any detail in the discussions below. Although BF term was historically proposed in an alternative scenario to the mass generation of vector gauge fields via Higgs, it can also be applied to a number of areas in condensed matter physics like theoy of topological superconductivity [50] and josephson junction array [51], effective theory of spin less p-wave superconductivity in 2d [52], hydrodynamic description of FQHE in 3d Weyl semimetals [53], hydrodynamic description of topological insulators [54, 55] etc. Let us now describe the two topological terms briefly.

1.3.1 Chern-Simons theory in 2+1d:

The Lagrangian of a pure Chern-Simons theory coupled to matter is given by [56]

$$\mathcal{L} = \frac{\lambda}{2}\varepsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda - A_\mu J^\mu \tag{1.97}$$

where A_μ is a Chern-Simons gauge field coupling to the matter current J^μ. We first note the fact that the Chern-Simons term is in general not gauge invariant but only upto a boundary term. It can also be shown using the equation of motion of the gauge field that when $J_\mu = 0$ the gauge field becomes a pure gauge i.e. $A_\mu = \partial_\mu\lambda$ which is not very interesting. But the dynamics becomes interesting when we couple J_μ with the CS gauge field as we shall see below. We have already seen in Sec. 1.2.2 that the equation of motion of CS field theory are given by

$$\varepsilon_{ij}\partial_i A_j = B = \frac{1}{\lambda}\rho, \qquad \varepsilon_{ij}E_j = \frac{1}{\lambda}J^i. \tag{1.98}$$

The first equation of Eq. (1.98) expresses the fact that the magnetic field is locally proportional to charge density. Thus through Chern Simons term flux is being bound to the charges, creating a charge-flux bound state. We shall show below how such a binding of magnetic flux to charges leads to fractional statistics of the charge-flux bound states. These particles obeying fractional statistics are called anyons. The second equation expresses that a charge current flows in the direction transverse to electric field- a feature which characterizes Hall effect. Both of these equation are of tremendous importance in expalining the FQHE at different filling factors.

For a non-relativistic distribution of point charges we can write $\rho(x) = e\sum_i \delta^2(x-x_i)$. So the magnetic field of the Chern-Simons gauge potential is given by $\varepsilon_{ij}\partial_i A_j = B(x) = \frac{e}{\lambda}\sum_i \delta^2(x-x_i)$. Thus for such a point charge distribution the magnetic field is only nonzero at points where charges exist and the value of corresponding flux is $\Phi = \frac{e}{\lambda}$. Thus each point particle will see another point particle as attached with a flux quantum $\frac{e}{\lambda}$. So when two such point charges rotate adiabetically, around one another the wave function of any of the particle changes by a Aharonov-Bohm phase $e^{\left(ie\oint \vec{A}\cdot d\vec{x}\right)} = e^{\left(i\,e^2/\lambda\right)}$. So if the particles are exchanged adiabetically the phase induced in ther wave function is $2\pi\Delta\theta = \frac{e^2}{\lambda}$ which can have any arbitraty value depending on the coefficient of Chern-Simons term λ. These point particles attached to flux quantum thus behave like anyons having fractional statistics as discussed above.

1.3.2 BF theory:

There can exist another topological term in 2+1d which is called mixed Chern-Simons term which involves two different gauge fields contracted with a Levi-Civita tensor which looks like : $\varepsilon^{\mu\nu\lambda}A_\mu\partial_\nu B_\lambda$. Such a term would naturally come when charge-vortex duality transformations are performed in 2+1d theories. A similar term can be constructed in 3+1d where one of the two gauge fields would be a 2-form field and is expressed as $\varepsilon^{\mu\nu\rho\lambda}B_{\mu\nu}F_{\rho\lambda}$, where $F_{\rho\lambda}$ is the field strength tensor for one-form field A_μ. This term is called a topological BF term. Topological BF theory gives rise to to mass of the associated gauge field A_μ. This mechanism can be understood by writing down the corresponding generating functional

$$\mathcal{Z} = \int \mathcal{D}A_\mu \mathcal{D}B_{\mu\nu} \exp\left(i\int d^4x \left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{M}{4}\varepsilon^{\mu\nu\rho\lambda}B_{\mu\nu}F_{\rho\lambda} + \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right)\right) \tag{1.99}$$

The mass generation of the gauge field A_μ can be shown in two different ways. One can determine the propagator or 2-point Green's function corresponding to A_μ diagramatically [57] or using functional techniques. One can find out that the propagator has devoloped a pole at $p^2 = M^2$ which suggests that the gauge field has become massive. Another way is to integrate out gauge field $B_{\mu\nu}$ by treating $\frac{M}{4}\varepsilon^{\mu\nu\rho\lambda}F_{\rho\lambda}$ as a current coupled to $B_{\mu\nu}$. This will leave us with

$$\mathcal{Z} \sim N \int \mathcal{D}A_\mu \exp\left(i\int d^4x \left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{M^2}{4}F^{\rho\lambda}\frac{1}{\Box}F_{\rho\lambda}\right)\right), \tag{1.100}$$

where $\frac{M^2}{4}F^{\rho\lambda}\frac{1}{\Box}F_{\rho\lambda}$ is the mass term for A_μ written in a gauge invariant form and N is a field independent term which have no contribution to generating functional. This can be easily understood by determining 2-point Green's function of A_μ from equation Eq. (1.100). Due to such property of mass generation, Abelian BF theories give rise to the Meissner effect and the London equation can also derived treating $J^\rho = \varepsilon^{\mu\nu\rho\lambda}H_{\mu\nu\lambda}$ as the conserved supercurrent and for such properties they are considered as topological theory of superconductivity. They are also used in many other condensed matter system as was discussed above.

Gauge theory of Josephson junction:

The mixed Chern-Simons term or BF term in 2+1d also appears in the theoretical description of Josephson junction array. A Josephson junction is a system where two superconducting bulk are separated by a normal or insulating bulk called a "weak link". This system has a special property that if $\Delta\phi$ be the phase difference of the order parameter between the two superconducting bulk then the current and voltage across the system obeys the following peculiar characteristics

$$I = I_c \sin(\Delta\phi), \quad V = \frac{\hbar}{2e}\frac{d\Delta\phi}{dt}. \tag{1.101}$$

Josephson junction arrays (JJA) are quadratic, two dimensional arrays of superconducting islands with spacing l and nearest-neighbour Josephson coupling of strength E_J. The capacitance of each islands to ground is C_0 and to nearest-neighbour is C. The Hamiltonian describing such a system is given by [58]

$$\mathcal{H} = \sum_n \left(\frac{1}{2}C_0 V_n^2\right) + \sum_{n,\mu}\left(\frac{1}{2}C\left(V_{n+\hat{\mu}} - V_n\right)^2 + E_J\left(1-\cos\left(N(\phi_{n+\hat{\mu}}-\phi_n)\right)\right)\right), \tag{1.102}$$

where n denotes the site of two dimensional array, V_n is the potential at the nth site and ϕ_n is the phase of order parameter at ith site. Here we shall try to show equivalence of this model to a lattice version of mixed Chern-Simon theory (or a BF term in 2+1d). For that we introduce a few notations and operators which are going to be useful in our derivation [51]. We assume that the Euclidean lattice on which we are placing our system have lattice spacing l_μ in 3 directions, in particular we consider $l_1 = l_2 = l$ in two spatial directions and l_0 in the Euclideanized time direction. Lattice sites will be denoted by n. The gauge fields will be associated to the links between two lattice sites n and $n+\hat{\mu}$ and will be denoted for example by: $A_{n,\mu}$, where $\hat{\mu}$ is the unit vector along the direction μ. The forward and backward derivative operators are defined as

$$d_\mu f(x) = \frac{f(x+l_\mu\hat{\mu}) - f(x)}{l_\mu}, \quad \hat{d}_\mu f(x) = \frac{f(x) - f(x-l_\mu\hat{\mu})}{l_\mu}. \tag{1.103}$$

We also introduce following finite difference operators

$$\Delta_\mu = l_\mu d_\mu, \quad \hat{\Delta}_\mu = l_\mu \hat{d}_\mu. \tag{1.104}$$

With the background introduced above we try to rewrite the second term in this Hamiltonian as

$$\begin{aligned} &\sum_{n,\mu}\frac{1}{2}C\left(V_{n+\hat{\mu}} - V_n\right)^2 \\ &= \sum_{n,\mu}\frac{1}{2}CV_{n+\hat{\mu}}\left(\Delta_\mu V_n - \Delta_\mu V_{n+\hat{\mu}}\right) \\ &= -\sum_{n,\mu}\frac{1}{2}CV_{n+\hat{\mu}}\hat{\Delta}_\mu\Delta_\mu V_{n+\hat{\mu}} \\ &= -\sum_{n,\mu}\frac{1}{2}CV_n\Delta^2 V_n, \end{aligned} \tag{1.105}$$

where $\Delta^2 = \hat{\Delta}_\mu\Delta_\mu$ is the Laplacian on lattice. So, we can write the Hamiltonian in the following form

$$\mathcal{H} = \sum_{n,\mu}\frac{1}{2}V_n(C_0 - C\Delta^2)V_n + E_J\left(1-\cos\left(N(\phi_{n+\hat{\mu}}-\phi_n)\right)\right). \tag{1.106}$$

It can be shown that the charge accumulated at any point in the JJA can be described by the following Poison's equation [58]

$$(C_0 - C\Delta^2)V_n = Q_n, \tag{1.107}$$

where Q_n is an integer multiple of Cooper pair charges: $Q_n = Nep_{0n}$, where p_{0n} is an integer defined on nth lattice point, N is is 2 for Cooper pairs. With Eq. (1.107) we can rewrite the Josephson Junction Array Hamiltonian as

$$\mathcal{H} = \sum_{n,\mu}\frac{1}{2}Q_n\frac{1}{C_0 - C\Delta^2}Q_n + E_J\left(1-\cos\left(N(\phi_{n+\hat{\mu}}-\phi_n)\right)\right). \tag{1.108}$$

In the limit $C_0/C << 1$ we have a Coulomb gas of charges in 2d interacting by a $\log(r)$ potential. In the following we shall consider the case for which $C_0 \to 0$. In this limit we rewrite the Hamiltonian as

$$\mathcal{H} = \sum_{n,\mu} \frac{1}{2} N^2 E_c p_{0n} \frac{1}{(-\Delta^2)} p_{0n} + E_J \left(1 - \cos\left(N(\phi_{n+\hat{\mu}} - \phi_n)\right)\right). \tag{1.109}$$

Now one can write down the generating functional for the Josephson Junction Array as

$$\mathcal{Z} = \prod_n \sum_{p_{0n}=-\infty}^{\infty} \int D\phi \exp(-S), \tag{1.110}$$

where the action S is given by

$$S = \sum_n -iNp_{0n}\Delta_0\phi_n + \sum_{n,\mu} \frac{1}{2} N^2 E_c p_{0n} \frac{1}{(-\Delta^2)} p_{0n} + E_J \left(1 - \cos\left(N(\phi_{n+\hat{\mu}} - \phi_n)\right)\right). \tag{1.111}$$

Here we have included the integration over Euclideanized time by extending the sum over a 3d Euclidean lattice and to compensate for the dimension of the corresponding measure we multiply each term by minimum lattice spacing along imaginary time direction l_0. The Hamiltonian of Eq. (1.109) is in conformity to the Hamiltonian of a 2 dimensional X-Y model and thus we shall proceed in exactly similar manner shown in Sec. 1.2.1 to bring vortices into the picture. Namely we first apply Villain transformation in the limit $l_0 E_J >> 1$:

$$\exp\left(l_0 E_J \left(1 - \cos\left(N\Delta_\mu\phi_n\right)\right)\right) \approx \sum_{v_{n,i}=-\infty}^{\infty} \exp\left(\frac{l_0 E_J}{2} N^2 \left(\Delta_i\phi_n + \frac{2\pi}{N} v_{n,i}\right)^2\right), \tag{1.112}$$

where $v_{n,i}$ is a new integer valued vector field. When this approximation is put into the generating functional we then have

$$\mathcal{Z} \approx \prod_{n,i} \sum_{v_{n,i}=-\infty}^{\infty} \sum_{p_{0n}=-\infty}^{\infty} \int D\phi \exp(-S), \tag{1.113}$$

where the action is given by:

$$S = \sum_n -iNp_{0n}\Delta_0\phi_n + \sum_{n,\mu} \frac{1}{2} N^2 E_c p_{0n} \frac{1}{(-\Delta^2)} p_{0n} + \frac{l_0 E_J}{2} N^2 \left(\Delta_i\phi_n + \frac{2\pi}{N} v_{n,i}\right)^2. \tag{1.114}$$

We now rewrite the partition function of Eq. (1.113) by linearizing the last term of Eq. (1.114) by introducing a real valued vector field $p_{n,i}$

$$\begin{aligned}
\mathcal{Z} &\approx \prod_{n,i} \sum_{v_{n,i}=-\infty}^{\infty} \sum_{p_{0n}=-\infty}^{\infty} \int \mathcal{D}\phi \mathcal{D}p_i \exp(-S), \\
S &= \sum_n -iNp_{0n}\Delta_0\phi_n + \sum_{n,\mu} \frac{1}{2} N^2 E_c p_{0n} \frac{1}{(-\Delta^2)} p_{0n} + \sum_{i,n} \frac{p_{n,i}^2}{2 l_0 E_J} - \sum_{i,n} iNp_{n,i}\left(\Delta_i\phi_n + \frac{2\pi}{N} v_{n,i}\right).
\end{aligned} \tag{1.115}$$

To treat $p_{n,0}$ in the same footing of $p_{n,i}$ we must make $p_{n,0}$ to be a real valued field. This can be done by introducing a new integer valued field $v_{n,0}$ and by using the Poison's summation formula

$$\sum_{v_{n,0}=-\infty}^{\infty} \exp\left(i2\pi v_{n,0} p_{n,0}\right) = \sum_{n=-\infty}^{\infty} \delta\left(p_{n,0} - n\right), n \in \mathcal{Z}. \tag{1.116}$$

With this new variable we have

$$\begin{aligned} \mathcal{Z} &\approx \prod_{n,\mu} \sum_{v_{n,\mu}=-\infty}^{\infty} \int \mathcal{D}\phi \mathcal{D}p_\mu \exp(-S), \\ S &= \sum_{n,\mu} \frac{1}{2} N^2 E_c p_{0n} \frac{1}{(-\Delta^2)} p_{0n} + \sum_{i,n} \frac{p_{n,i}^2}{2l_0 E_J} - \sum_{\mu,n} iN p_{n,\mu} \left(\Delta_\mu \phi_n + \frac{2\pi}{N} v_{n,\mu}\right). \end{aligned} \tag{1.117}$$

At this point, let us write the vector field $v_{n,\mu}$ as a sum of a lognitudinal part and a transverse part:

$$v_{n,\mu} = \Delta_\mu m_n + \Delta_\mu \alpha + \sum_{\nu\lambda} \varepsilon_{\mu\nu\lambda} \Delta_\nu \psi_{n,\mu}, \tag{1.118}$$

where first two terms represent the longitudinal part (or irrotational part) and the last term represent the transverse part (or the solinoidal part). We note the fact that the vector field $v_{n,\mu}$ is an integer valued field. As we define $\psi_{n,\mu}$ to be a real valued field and m_n to be an integer valued field we needed to add a real valued vector $\Delta_\mu \alpha$ to make the sum an integer. Also let us define $q_{n,\mu} = \sum_{\nu,\lambda} \varepsilon_{\mu\nu\lambda} \Delta_\nu v_{n,\lambda}$ which must be an integer valued field. Now one can easily see that the field $q_{n,\mu}$ is constrained by the relation:

$$\sum_\mu \Delta_\mu q_{n,\mu} = 0. \tag{1.119}$$

Also we can write the term $\sum_{\mu,n} p_{n,\mu} \Delta_\mu \phi_n$ in the action in Eq. (1.117) as $\sum_{\mu,n} \Delta_\mu p_{n,\mu} \phi_n$ and therefore integration over fields ϕ_n, defined on each lattice points on a 3d Euclidean lattice, give us delta functions: $\delta(\sum_\mu \Delta_\mu p_{n,\mu})$ which implement the constraint for each n:

$$\sum_\mu \Delta_\mu p_{n,\mu} = 0. \tag{1.120}$$

These two constraints expressed by Eq. (1.119) and Eq. (1.120) can be solved by writing $q_{n,\mu}$ and $p_{n,\mu}$ in terms of a integer valued vector field $a_{n,\mu}$ and a real valued vector field $b_{n,\mu}$ as:

$$\begin{aligned} q_{n,\mu} &= \sum_{\nu,\lambda} \varepsilon_{\mu\nu\lambda} \Delta_\nu a_{n,\lambda}, \quad a \in \mathcal{Z}, \\ p_{n,\mu} &= \sum_{\nu,\lambda} \varepsilon_{\mu\nu\lambda} \Delta_\nu b_{n,\lambda}, \quad b \in \mathcal{R}. \end{aligned} \tag{1.121}$$

Thus substituting these equations into our generating functional we get finally

$$\begin{aligned} \mathcal{Z} &\approx \prod_{n,\mu} \sum_{Q_n=-\infty}^{\infty} \int \mathcal{D}a_\mu \mathcal{D}b_\mu \mathcal{D}p_\mu \exp(-S), \\ S &= \sum_{n,\mu} \frac{1}{2} N^2 E_c p_{0n} \frac{1}{(-\Delta^2)} p_{0n} + \sum_{i,n} \frac{p_{n,i}^2}{2l_0 E_J} - 2\pi i \sum_{n,\lambda,\nu,\mu} \varepsilon_{\lambda\nu\mu} b_{n+\hat{\nu},\lambda} \Delta_\nu a_{n,\mu} + \sum_{n,\mu} 2\pi i Q_n a_{n,\mu}, \end{aligned} \tag{1.122}$$

where the last term is added by using Poisson's summation formula for making $a_{n,\mu}$ a real valued field. The last equation is the action we wanted to obtain which contains the required lattice version of mixed Chern-Simons term expressed by the third term in the action. In this representation $\sum_{\nu,\lambda} \varepsilon_{\mu\nu\lambda}\Delta_\nu b_{n,\lambda}$ represents the conserved three-current of charges, While $\sum_{\nu,\lambda} \varepsilon_{\mu\nu\lambda}\Delta_\nu a_{n,\lambda}$ represents the conserved 3-current of vortices. Thus the mixed CS term expresses a coupling between charges and vortices.

Spin-gauge theory:

In 2+1d the gauge fields involved in a mixed Chern-Simons term couples to particle like matter current through minimal coupling scheme. In 3+1d, although electromagnetic vector field A_μ couples to a conserved particle current, $B_{\mu\nu}$ couples naturally only to extended objects like strings (or vorticity). We have already encountered such facts in Eq. (1.95). But one can also ask if there can be any coupling between $B_{\mu\nu}$ and particle like matter current, in particular to a fermionic current. While constructing any such term we have to keep in mind that action of $B_{\mu\nu}$ field has a different type of gauge symmetry, namely the vector gauge symmetry given by: $B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu\Lambda_\nu - \partial_\nu\Lambda_\mu$, where Λ_μ is a vector field. Thus the term in which fermions couples to 2-form field must be invariant under such a gauge transformation. Also there is the fact that the 2-form field naturally couples to extended objects like 1d strings, so the fermionic current which couples to this 2-form field must have non locality built into it. Considering all such issues a coupling between 2-form field and fermionic matter was proposed in [59] which was symmetric under P and T transformation and is given by the following term

$$S_{int} = \int d^4x \left(B_{\mu\nu}\varepsilon^{\mu\nu\rho\lambda}\frac{1}{\Box}\partial_\rho\left(\bar{\psi}\gamma_\lambda\psi\right)\right) \tag{1.123}$$

It was also shown in that paper that such a term generates a BF term when radiative corrections due to fermionic loops are considered. As was shown previously a BF theory, thus induced in the action, may generates mass for the associated gauge field and hence corresponds to some superconducting phase. Also it was later shown in [60] if one calculate potential between two fermions which are nearly static one gets a linearly rising potential $V(r) \sim r$ confining two fermions into a pair. Such a potential is often encountered in theories of quark confinement and thus is very important in that context. It is also possible that such pairs are realized in low energy condensed matter systems like some unconventional superconductivity. As the theory is defined to be a non-local theory, one can expect to find a local theory which in some low energy phase will give rise to this non local coupling. Thus this new term may have important application to low energy phase of QCD and to unconventional superconductivity. In this thesis we shall mostly be concerned about finding out the local theory from which this term emerges and finding out it's implications for low energy phenomenology.

1.4 Statistical transmutation in condensed matter:

One of the most important concepts in modern condensed matter physics and also one of the major concern of our thesis is statistical transmutation of particles. The fundamental degrees of freedom in condensed matter systems are electrons which are fermionic in nature i.e. they obey Fermi-Dirac distributions. All of atomic physics and many-body quantum theories like Hartree-Fock and density functional theory(DFT) are built on Pauli's exclusion principle which is deeply associated with the fermionic nature of electronic excitations. But there exists low energy physical systems which can not be described in terms of these electronic degrees of freedom but rather some collective degrees of freedom composed of electrons. Below we are going to dicuss two examples of such statistical transmutation realized in physical systems like superconductors and fractional quantum hall effect.

1.4.1 Bosonization through fermion pairing:

Examples of such systems are superconductivity appearing in metals at very low temperature where the collective behaviour of the system can be explained in terms of Cooper pairs, which approximately can be treated as spin-0 bosons (represented by scalar fields) in the case where total spin and orbital angular momentum of the cooper pairs are taken to be zero. Thus in such systems there exists a statistical transmutation from fermions to spin-0 bosons[4]. Such statistical transmutation from fermionic to bosonic statistics are referred to as bosonization in most literature. One can also obtain a bosonic theory out of a underlying fermionic theory by decoupling in a particular channel through Hubbard-Stratonovich transformation- one can be the Cooper channel where the bosonic field represent a pair of electrons with opposite spin, another can be density channel in which bosonic field represent charge density wave [62]. Below we shall discuss a direct method of bosonization starting from a theory of interacting fermions in 1 dimension where the bosons will represent fermionic density excitations. We start with the following Hamiltonian

$$\hat{H} = \sum_k a_k^\dagger \left(\frac{k^2}{2m} - E_F \right) a_k + \frac{1}{2L} \sum_{k,k',q\neq 0} V(q) a_{k-q}^\dagger a_{k'+q}^\dagger a_{k'} a_k. \tag{1.124}$$

Physical systems which can be modeled by such Hamiltonian are the low energy bands of a carbon nanotube, quasi one dimensional semiconducting wires, edge modes in a quantum Hall system, stripe phases in high temperature superconductors etc. One dimensional systems possess many peculiar properties which are not shared by higher dimensional systems. As an example, in one dimension electrons are constrained to stay along a line, in presence of inter electron interaction, they must push each other to minimize the energy of the configuration. Such interactions result in formation of density waves as we shall see below. In higher dimensions however, electrons can avoid each other by moving away from one another in the available space. This can also be understood by noticing the configuration of Fermi sphere in one dimension. In one dimension, the Fermi sphere is the interval $-k_F < k < k_F$, and the Fermi surface thus becomes two discrete points $k_F, -k_F$. In comparison, Fermi surfaces in higher dimensional material systems are connected surfaces. For this, in one dimension there exists very few states in which particles can shift in presence of interaction. We shall show below that such a one dimensional electron gas is equivalent to a gas of bosons which represents the density excitations in the original electron systems.

As we have already discussed the relevant excitations will be those which are excited above the Fermi momentum $k_F, -k_F$. For this one may expand the energy spectrum

[4] However, there can be transmuatation from purely fermionic theory to spin-1 bosonic theory. See for example the work by Fradkin and Schaposnik. [61]

around the fermi points as

$$E_k = \frac{k^2}{2m} - E_F = \frac{(p+k_F)^2}{2m} - E_F \simeq v_F p, \quad v_F = \frac{k_F}{m}. \tag{1.125}$$

As all the states for which $p = k - k_F < 0$ are filled up the corresponding terms in Hamiltonian will not contribute to the ground state. Hence the non interacting Hamiltonian can be written ignoring those terms as

$$H_0 = \sum_p v_F p \left[a^\dagger_{K_F+p} a_{k_F+p} + a^\dagger_{-K_F-p} a_{-k_F-p} \right]. \tag{1.126}$$

We rewrite the creation and annihilation operator as $a^\dagger_{k_F+p} = a^\dagger_{Rp}$ which creates a right moving particle and $a^\dagger_{-k_F-p} = a^\dagger_{Lp}$ which creates a left moving one. Writing the non interacting Hamiltonian in terms of these new notations we have

$$H_0 = \sum_{s,p} v_F p \left[a^\dagger_{sp} a_{sp} \right], \tag{1.127}$$

where $s = R, L$ represents the right or left moving particles. The sum over p is restricted by a cut off momentum Λ. From now on we shall only consider quantum states over the Fermi points and so all the momentum variable that we would use in the subsequent discussions will be taken above Fermi points. This makes Fermi momentum the new origin of one dimensional momentum space. Next we define density operator as

$$\rho = \sum_k a^\dagger_{sk+p} a_{sk}. \tag{1.128}$$

We shall try to determine the commutation relation of such density operators below which will again be used in the definition of effective bosonic commutaion ralations. Thus we write

$$[\hat{\rho}_{sq}, \hat{\rho}_{s'q'}] = \sum_{k,k'} [a^\dagger_{s\,k+q} a_{s\,k} a^\dagger_{s'\,k'+q'} a_{s'\,k'} - a^\dagger_{s'\,k'+q'} a_{s'\,k'} a^\dagger_{s\,k+q} a_{s\,k}]. \tag{1.129}$$

We now use fermionic anti-commutation relation $\{a^\dagger_{s\,k}, a_{s'\,k'}\} = \delta_{s,s'}\delta_{k,k'}$ in the above equation and get

$$[\hat{\rho}_{sq}, \hat{\rho}_{s'q'}] = \delta_{s,s'} \sum_k [a^\dagger_{s\,k+q} a_{s'\,k-q} - a^\dagger_{s'\,k'+q'} a_{s'\,k'-q'}]. \tag{1.130}$$

As we can understand that the other two terms like $a^\dagger_{s\,k+q} a^\dagger_{s'\,k'+q'} a_{s\,k} a_{s'\,k'}$ that would arise while using the fermionic anti-commutation relation would become zero as we take expectation of the commutator in the ground state. For this we have dropped those two terms. In practice we use such commutation relation in determination of various physical quantities where expectation values of the physical quantities are determined in a particular ground state. Noticing this fact we want to simplify the right hand side of the Eq. (1.130) by taking average in the ground state of the system $|\Omega\rangle$ and then we get

$$\begin{aligned} [\hat{\rho}_{sq}, \hat{\rho}_{s'q'}] &= \delta_{s,s'} \sum_k \langle \Omega | [a^\dagger_{s\,k+q} a_{s\,k-q} - a^\dagger_{s\,k+q+q'} a_{s\,k}] | \Omega \rangle \\ &= \delta_{s,s'} \delta_{q,-q'} \sum_k \langle \Omega | (n_{s\,k+q} - n_{s\,k}) | \Omega \rangle , \end{aligned} \tag{1.131}$$

where $n_{s\,k} = a^\dagger_{s\,k} a_{s\,k}$ is the diagonal part of the density operator which appears due to taking expectation value in the ground state. If the sum over momentum k extends over all values of momentum the above sum will become zero. This is not the case here as the momentum has a cutoff Λ. To evaluate this sum correctly we remember the fact that we have set the Fermi momentum to be zero. Thus all the states for which $k < 0$ are filled up and above that level states are empty. We choose $s = R$ and redefine the momentum variable for the first sum to be $t = k + q$. Thus we get

$$\begin{aligned}
[\hat\rho_{Rq}, \hat\rho_{Rq'}] = \delta_{q,-q'} \Bigg[&\sum_{-\Lambda+q\le t<0} \langle\Omega|n_{R\,t}|\Omega\rangle + \sum_{t\ge 0} \langle\Omega|n_{R\,t}|\Omega\rangle \\
&- \sum_{-\Lambda\le k<0} \langle\Omega|n_{R\,k}|\Omega\rangle - \sum_{k>0} \langle\Omega|n_{R\,k}|\Omega\rangle \Bigg].
\end{aligned} \tag{1.132}$$

We note that the second and fourth terms will becomes zero because $n_{k=0}\,|\Omega\rangle = 0$ while the first and third term will contribute and we shall get

$$[\hat\rho_{Rq}, \hat\rho_{Rq'}] = \frac{L}{2\pi}\delta_{q,-q'} \left[\int_{-\Lambda+q}^{0} dk - \int_{-\Lambda}^{0} dk \right] = -\frac{qL}{2\pi}\delta_{q,-q'}. \tag{1.133}$$

Similar expression can be derived for the left movers ($s = L$) as well. So in general we can write

$$[\hat\rho_{sq}, \hat\rho_{s'q'}] = -\delta_{q,-q'}\delta_{s,s'}\frac{qL}{2\pi}. \tag{1.134}$$

Now we shall try to rewrite the initial Hamiltonian Eq. (1.124) in terms of these newly introduced density operators ρ. First we focus on the interaction term

$$\begin{aligned}
H_{int} &= \frac{1}{2L}\sum_{k,k',q} V(q) a^\dagger_{k-q} a^\dagger_{k'+q} a_{k'} a_k \\
&= \frac{1}{2L}\sum_{s\,q} \Big[2V(q\sim 0)\hat\rho_{s\,q}\hat\rho_{s\,-q} + 2V(q\sim 2k_F)\hat\rho_{s\,q}\hat\rho_{\bar s\,-q} \Big],
\end{aligned} \tag{1.135}$$

where in the last step we have used the commutation relations derived above. Writing $2V(q\sim 0) = g_4$ and $2V(q\sim 2k_F) = g_2$ we rewrite the interaction Hamiltonian as

$$H_{int} = \frac{1}{2L}\sum_{s\,q} \Big[g_4\hat\rho_{s\,q}\hat\rho_{s\,-q} + g_2\hat\rho_{s\,q}\hat\rho_{\bar s\,-q} \Big], \tag{1.136}$$

where we have denoted the alternative chirality of s by $\bar s$. Now we shall define creation and annihilation operators for collective bosonic excitations using the fermionic density operators as follows:

$$b_q = n_q\hat\rho_{Lq}, \qquad b^\dagger_q = n_q\hat\rho_{L-q} \tag{1.137}$$

$$b_{-q} = n_q\hat\rho_{R-q}, \qquad b^\dagger_{-q} = n_q\hat\rho_{Rq}. \tag{1.138}$$

With these definition one can rewrite the interacting part of the Hamiltonian as follows

$$H_{int} = \sum_q \frac{q}{4\pi}[g_4 b_q b^\dagger_q + g_4 b^\dagger_{-q} b_{-q} + g_2 b_q b_{-q} + g_2 b^\dagger_{-q} b^\dagger_q]. \tag{1.139}$$

We also need to write the non-interacting part of the original system in terms of those bosonic operators. We shall take a indirect route for obtaining a kinetic term for the bosonic Hamiltonian. As in the second quantized description, properties of an operator is fixed by commutation relations. We shall try to look for an operator H_0' which obeys the same commutation relations with bosonic operators as the original non-interacting Hamiltonian H_0. One can easily verify that $[H_0, \hat{\rho}_{sq}] = qv_F\sigma_s\hat{\rho}_{sq}$. One can also show that the same relation is obeyed by the operator

$$H_0' = \frac{2\pi v_F}{L}\sum_{qs}\hat{\rho}_{s\,q}\hat{\rho}_{s\,-q} = \sum_q qv_F\left[b_q b_q^\dagger + b_{-q}^\dagger b_{-q}\right]. \tag{1.140}$$

Thus we obtain the total Hamiltonian in terms of bosonic creation and annihilation operators as

$$H = \sum_q q\left[\frac{g_4}{2\pi} + v_F\right]b_q b_q^\dagger + \left[\frac{g_4}{2\pi} + v_F\right]b_{-q}^\dagger b_{-q} + g_2 b_q b_{-q} + g_2 b_{-q}^\dagger b_q^\dagger. \tag{1.141}$$

Thus we see that a one dimensional electron gas can be expressed as a gas of bosonic cvollective particles whose dynamics is given by the above Hamiltonian. This is an example of statistical transmutation or bosonization. In the next section we shall discuss statistical transmutation in one higher spatial dimensions by a completely different method. The importance of writing the original systems in terms of these effective bosonic particles can be realized once we write the above Hamiltonian in the Nambu-Gor'kov basis $\begin{pmatrix} b_q^\dagger \\ b_{-q}\end{pmatrix}$. In this basis the Hamiltonian reads

$$H = \begin{pmatrix} v_F + \dfrac{g_4}{2\pi} & g_2 \\ g_2 & v_F + \dfrac{g_4}{2\pi}\end{pmatrix}. \tag{1.142}$$

Diagonalizing this Hamiltonian one gets the following free theory of bosons

$$H = \sum_q qv_b\tilde{b}_q^\dagger\tilde{b}_q, \quad v_b = \frac{1}{2\pi}[(2\pi v_F + g_4)^2 - g_2^2]. \tag{1.143}$$

Thus we have successfully converted the interacting theory of electrons to a non interacting theory of bosons which now can be analysed very easily and the behaviour of the system in the low energy limit can be predicted.

1.4.2 Bosonization by flux attachment:

There can be a completely different type of bosonization than via fermion pairing- which can be realized through flux attachment [39, 40]. In this case one starts from a fermionic system exposed to an external magnetic field. Transformation of fermionic wave functions by a unitary operator, which includes a multivalued scalar, leads to a transformation of the Hamiltonian of the system such that the transformed wave function behave like bosonic wave function coupled to a gauge field. This new gauge field is the statistical gauge field, which in second quantized theory depends on the density of the bosons. Thus a set of fermions in an external magnetic field may be represented as a set of bosons coupled minimally to a density dependent gauge field. This gauge field is actually the Chern-Simon gauge field and through the Chern-Simons term its flux is attached to the particles. We are going to explore a new way of attaching flux to particles having spin in one of our subsequent chapters in this thesis. Below we shall describe this scheme of flux attachment in details which would be helpful for comparison with our latter discussions.

Let us start from a system of electrons confined to move on a 2 dimensional plane- a two dimensional electron gas- exposed to external magnetic field perpendicular to the surface.

$$H = \frac{1}{2m}\sum_n [\vec{p_n} - \frac{e}{c}\vec{A}(x_n)]^2 + \sum_i eA_0(x_n) + \sum_{n<m} V(x_n - x_m), \tag{1.144}$$

where $\vec{A}(x)$ is the vector potential corresponding to external magnetic field which in symmetric gauge takes the form: $A_i = \frac{1}{2}B\varepsilon_{ij}x_j$. A_0 is the external scalar potential producing the electric field $E_\mu = -\partial_\mu A_0$ and $V(x)$ is the interaction potential energy between two electrons separated by $\vec{x}$ and in this problem it is the Coulomb potential. The indices $n, m = 1, 2, ...$ represents electrons, $i, j = 1, 2$ are spatial indices, and $\mu = 0, 1, 2$ are space-time indices. We also assume that the external magnetic field is so strong that electron magnetic moments are aligned along the magnetic field i.e. the system is completely spin polarized. In such a situation only the spatial part of the wave function of the many electron system would be important. So this spatial wave function $\Psi(x_1, x_2, ..)$ must be totally anti-symmetric according to Pauli's exclusion principle. The Schrödinger equation for this problem is given by

$$H\Psi(x_1, x_2, x_3, ...) = E\Psi(x_1, x_2, x_3, ...). \tag{1.145}$$

This equation along with the anti-symmetry of many electron wave function defines the quantum eigenvalue problem. In the following discussion we shal show that this problem is equivalent to a problem of bosons coupled to a new gauge field a_α termed as statistical gauge field. The bosonic problem is defined by the following Schrödinger equation

$$\begin{aligned} &H'\phi(x_1, x_2, ...) = E'\phi(x_1, x_2, x_3, ...), \\ &H' = \frac{1}{2m}\sum_n [\vec{p_n} - \frac{e}{c}\vec{A}(x_n) - \frac{e}{c}\vec{a}(x_n)]^2 + \sum_n eA_0 + \sum_{n<m} V(x_n - x_m). \end{aligned} \tag{1.146}$$

The new gauge potential is given by

$$\vec{a}(x_n) = \frac{\phi_0}{2\pi}\frac{\theta}{\pi}\sum_{n\neq m}\vec{\nabla}\alpha_{nm}, \tag{1.147}$$

where $\phi_0 = \frac{hc}{e}$ is the flux quantum, θ is a parameter to be adjusted according to our requirement and α_{nm} is the angle subtained by the vector connecting nth and mth particle and a reference direction. The wave function $\phi(x_1, x_2, ...)$ is bosonic according to our consideration and hence must be symmetric under exchange of particles. Let us now transform the bosonic wave function as following

$$\tilde{\phi}(x_1, x_2...x_N) = U\phi(x_1, x_2...x_N); \quad U = \exp\left(-i\frac{\theta}{\pi}\sum_{n<m}\alpha_{nm}\right) \tag{1.148}$$

To see what happens to the Schrödinger equation for bosonic particles given by Eq. (1.146), we shall multiply both sides of this equation by the unitary operator U. Writing $\phi = U^\dagger\tilde{\phi}$ we find

$$UH'U^\dagger\tilde{\phi}(x_1, x_2, ...) = E'\tilde{\phi}(x_1, x_2, x_3, ...). \tag{1.149}$$

Thus we see that the Hamiltonian of the system changes to $H' \to UH'U^{\dagger}$. To see the effect of this transformation we note that $U\left(\vec{p_i} - \frac{e}{c}\vec{A}(x_i) - \frac{e}{c}\vec{a}(x_i)\right)U^{\dagger} = \left(\vec{p_i} - \frac{e}{c}\vec{A}(x_i)\right)$, where we have used the fact that $\vec{p_i}$ is a differential operator and also have used the definition of the gauge field $\vec{a}$. Using this observation we can easily see that

$$UH'U^{\dagger} = \frac{1}{2m}\sum_n [\vec{p_n} - \frac{e}{c}\vec{A}(x_n)]^2 + \sum_i eA_0(x_n) + \sum_{n<m} V(x_n - x_m). \tag{1.150}$$

We see that right hand side of Eq. (1.150) is exactly same as the Hamiltonian of the original problem of two dimensional electron gas. To check how the new many-particle wave function $\tilde{\phi}$ behaves under the exchange of coordinates of two particles let us exchange ith and jth corordinate. Under the operation we find

$$\tilde{\phi}(x_1, ...x_j...x_i..x_N) = \exp\left(-i\frac{\theta}{\pi}(\alpha_{12} + \alpha_{13} + ...\alpha_{ji} + ...)\right)\phi(x_1, ..x_j..x_i...x_N) \tag{1.151}$$

As ϕ is symmetric under coordinate exchange we have: $\phi(x_1, ..x_j..x_i...x_N) = \phi(x_1, ..x_i..x_j...x_N)$. But we note that according to the definition of the angle α_{ij} we must have $\alpha_{ji} = \pi + \alpha_{ij}$. Using these facts we see that

$$\tilde{\phi}(x_1, ...x_j...x_i..x_N) = \exp(-i\theta)\exp\left(-i\frac{\theta}{\pi}(\alpha_{12} + \alpha_{13} + ...\alpha_{ij} + ...)\right)\phi(x_1, ..x_j..x_i...x_N). \tag{1.152}$$

Now we see that for $\theta = (2n+1)\pi$ the multiparticle wave function $\tilde{\phi}$ changes sign under particle exchange. So the wave function $\tilde{\phi}$ must be totally antisymmetric under particle exchange as the original multielectron wavefunction. Thus this transformation leads to a totally anti-symmetric wave function obeying the Schrödinger equation

$$\left[\frac{1}{2m}\sum_n [\vec{p_n} - \frac{e}{c}\vec{A}(x_n)]^2 + \sum_i eA_0(x_n) + \sum_{n<m} V(x_n - x_m)\right]\tilde{\phi}(x_1, ...x_N) = E'\tilde{\phi}(x_1, ...x_N). \tag{1.153}$$

The left hand side of Eq. (1.153), being equal to left hand side of Eq. (1.145), the eigen energy spectrum of both equations must also be same i.e $E = E'$. Thus from the above discussions, we can conclude that the bosonic problem defined by Eq. (1.146) is completely equivalent to the original many-electron problem for $\theta = (2n+1)\pi$. This is one of the schemes of statistical transmutation which says that in 2+1 dimension a fermionic problem is completely equivalent to a bosonic problem with a new gauge potential coupled to it. The transformation involves a multivalued scalar α_{ij} and hence is called a “singular gauge transformation”. We have already discussed about the transformations for bosonic field operators in second quantized formalism in Sec. 1.2.2 and have shown how such transformation leads to a density dependent gauge field

$$a_i = k\int d^2x' \partial_i^x \Theta(x - x')\rho(x') = k\epsilon_{ij}\int d^2x' \frac{x_j - x'_j}{|\vec{x} - \vec{x}'|^2}\rho(x'). \tag{1.154}$$

This gauge potential is equivalent to the one (i.e. $\vec{a}$) discussed in this section when ρ is written for a discrete mnay particle distribution. For this reason such a gauge field is called a statistical gauge field. As we have discussed in Sec. 1.2.2, the dynamics of the statistical field is governed by Chern-Simons term.

Thus one can conclude that the theory of two dimensional electron gas coupled to an external vector potential (magnetic field) is equivalent to a gas of bosons coupled to a statistical gauge field (in addition to the external gauge field) obeying the equation of motion of Chern-Simons gauge theory. As we have already saw, what Chern-Simons term does is to attach odd number of flux quantums to each electron and convert them into bosonic quasi-particles obeying bosonic commutation rule. Thus flux attachment becomes the central idea for statistical transmutation in this theory. We shall explore more into this idea of “flux attachnment” in this thesis and shall try to propose a mechanism for flux attachment in the contecxt of dual boson-fermion model.

1.4.3 Brief Outlines to the Thesis:

In this work we are going to apply the charge vortex duality to a system consisting of charged bosons and fermions interacting via electromagnetic gauge field. Such a relativistic boson fermion mixture can be thought of as a relativistic generalization of the Friedberg-Lee model of high temperature cuprate superconductors. Although first proposed to explain qualitative features of high temperature superconductors, Friedberg-Lee model was later adopted in various scenarios like BCS-BEC crossover, superconductor to insulator transition etc. The only difference in our case is that we have neglected the direct cubic coupling between fermions and bosons.

In Chap. 2 we show how the charge-vortex duality being applied to above mentioned system leads to the electron-vortex interaction and how such an interaction affects charge-charge interaction potential. Through application of duality we wanted to bring vortices into the picture where fermions are also present. In this treatmnent we have assumed that presence of fermions will not alter the Nielsen-Olesen vortex solutions much which is implemented by taking the same decomposition of the phase in regular and singular parts as in absence of the fermions. Thus dualization of the scalar field is done same as before, but the fermions are going to be involved into dualizing the the Maxwell term. In doing so we end up in a phase where vortices (or strings) interact with a non-local fermionic current via 2-form field $B_{\mu\nu}$. This interaction differs from the Aharanov-Bohm interaction between charge particles and vortices discussed in literature [63]. The $0i$th component of the non-local fermionic current actually expresses the spin moment density. Thus this interaction in the static case corresponds to spin-string interaction. Exploration of this interaction in the contexts of low energy systems of different dimensionality will be the main motivation of the rest part of this thesis.

First we shall explore the consequences of such an interaction on charge charge interaction potential by calculating fermionic loop correction of photon propagator. The calculation is done using familiar functional techniques. This calculations give rise to a $B \wedge F$ term with cutoff dependent coefficient. The cutoff momentum would be of the order of inverse coherence length of superconductor: $\xi_\lambda = 1/\sqrt{\lambda}v$. This $B \wedge F$ term along with the gauge invariant mass term of 2-form field will give rise to a coulombic term with a high dielectric constant in the charge-charge interaction potential. The reason for this can be found out by comparing the quantum corrected effective theory with the disorder field theory of Kleinert [64]. Thus vortex-electron interaction leads to an effective theory similar to the disorder theory of Kleinert which may be quite important in the above mentioned contexts.

In Chap. 3 we have shown that the electron-2-form interaction can lead to attachment of spin current to vorticity in 3+1 dimension and tried to argue in various ways that such attachment may lead to a configuration where two electrons are attached to each other by means of two flux tubes. Due to the formation of such a configuration a linear potential may act between electrons.

We had mentioned before that when the coupling between $B_{\mu\nu}$ and fermions was proposed it was shown that the fermions interact via a linearly rising static potential. As in the dual model such an interaction arises naturally from the dualization, one can ask whether there exists any phase of the system where linearly rising potential can be found between two static fermions. To answer this question we first note that electrons attaches to vortices due to their non-zero spin magnetic moment. Due to such an attachment dipole-dipole interaction between two electrons become screened at large distance.

But when two such flux attached electrons approach each other and reach with in a distance $2\xi_e \sim 1/ev$ (penetration depth) to each other their magnetic field lines starts to interact because within penetration depth magnetic field is unscreened. In this situation, as magnetic field inside a superconductor must pass through vortices, magnetic field lines produced by spin magnetic moment colease to flux tubes and end up on another electron. Thus the above mentioned configuration may appoear and a thus a linear potential between electrons acts. This intuitive picture can be supported by noting that the momentum corresponding the scale $\xi_e = 1/ev$ is $\sim ev$ and thus this should be the minimum exchange momentum of the electrons. If we now expand the propagator of the $B_{\mu\nu}$ field with the approximation $k^2 > (ev)^2$ one finds an interaction potential $v(k) \sim 1/k^4$ which gives rise to a potential $v(r) \sim r$ in the limit $\lambda \to \infty$, $e \to 0$.

The above argument apply only when $\xi_e >> \xi_\lambda$ and thus would fail in the limiting case $\xi_e \sim \xi_\lambda$ i.e. when the vortices act as 1 dimensional strings. However in this limit as well we have found arguments which lead to confining potential. Here we note that the dual theory is in fact a theory of classical strings interacting via a massive 2-form gauge field where strings are represented by world sheets swept by them. One can now generalize the classical string action to field theoretic action of strings where we assign a string field, defined on the space of curves or loops. In such a theory the mass of the 2-form can be explained if we assume that the strings are in a condensed phase where the 2-form become massive via a "stringy Higgs mechanism". But such a string field theory may also contain a normal state(or a "false vacuum") where 2-form becomes massless. In such a normal state the theory would be described by the dual action excluding the mass term of 2-form field. Thus the dual theory in such a normal vacuum of string field would give rise to electron-electron linear interaction potential.

In Chap. 4 we try to explore consequences of the electron-vortex interaction in the context of physics of two spatial dimensions. A realistisc situation where our model may be relevant is the interface between a superconductor and a topological insulator(TI) or graphene or similar materials hosting a 2 dimensional electron gas. To proceed we have dimensionally reduced the dual Lagrangian. The dimensionally reduced theory contains vortices which are now point like objects and interact with a non-local fermion current confined in two dimensional plane via a one-form field. To achieve flux attachment from our theory we compare our theory with Zhang-Hansson-Kivelson theory of FQHE [6,65] and define a new gauge field C_μ whose curl is the vorticity vector Σ_μ in the reduced theory. We then obtain the effective theory of C_μ gauge field and show that equation of motion of C_μ gives rise to the same flux attachment condition in some static approximation as obtained before. Flux attachment is in general obtained from the Chern-Simons term as we have seen before. As we obtain it here as well, it suggests that the flux-spin composites in this case may obey fractional statistics. To show this we derive the non-relativistic theory of fermions coupled to the gauge field C_μ. When flux-spin composites are taken around each other, the non-relativistic theory suggests that an Aharanov-Bohm phase factor is induced in the wave function. This phase may become arbitrary when the factor $\kappa = 1/\sqrt{L}$, where L is extension of the fields along z direction, takes arbitrary values. This may alter the statistics of the flux attached electrons making them anyons. Thus we show anyons can be realized in such a system.

We end our discussion by summarising all the results of our investigations shaortly and mentioning possible direction of future research in Chap. 5.

Chapter 2

Emergent Electron-Vortex Interaction from Dualization

2.1 Introduction

Vortices and vortex lines appear as solutions in the field theoretic description of many physical systems, from quantized vortices in superfluid Helium [66–69] and Bose-Einstein condensates to Abrikosov lattices in type-II superconductors [70, 71], all of which have been observed. They also appear in field theories which describe high energy physics, e.g. as cosmic strings which appear in many gauge theories including grand unified theories [72–75], or as color flux tubes which are conjectured to appear in non-Abelian gauge theories such as QCD, leading to color confinement [76–78]. Here we shall be dealing only with vortices appearing in condensed matter systems. In particular we will be interested in systems where charged fermions coexists with vortices, and therefore an interaction of vortices with fermions can appear. As appearance of vortices is typically seen in scalar matter (certain exceptions exist [79]), we will be looking into such systems where a mixture of bosons and fermions would exist. Below we give certain famous examples of such systems.

2.1.1 Boson Fermion model and its applications:

A mixture of interacting bosons and fermions at very low temperatures contains a rich variety of physical phenomenon. In particular it becomes immensely important in the context of high T_C cuprate superconductor as it is shown by various authors that different versions of such a model can qualitatively explain the rich phase diagram of such superconductors.

The phenomenon of superconductivity at high temperature has remained mysterious since its discovery in cuprates [80] -the Bardeen-Cooper-Schrieffer (BCS) description of low-temperature superconductivity [35, 81, 82] can not explain it. The typical features of a high temperature superconductor (HTS) are expressed through a phase diagram [83,84] which looks like Fig. 2.1. In the absence of doping, the cuprate material remains anti-ferromagnetic (AFM) and insulating. As doping is increased, anti-ferromagnetism does not persist and at small doping concentration and below a temperature T^* a gap opens up in the electronic energy spectrum, which is called a pseudo gap [85–87]. As doping is increased further, superconductivity (SC) starts to appear beyond this gap. The presence of a pseudo gap above T_c and very small coherence length (~ 10 Å) [88] indicates the

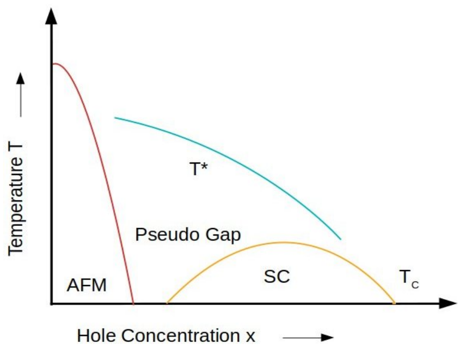

Figure 2.1: Phase Diagram Of A Hole Doped High T_c Superconductor

formation of localized bosonic pairs of fermions (preformed pairs [83,89,90]) below T^* and their condensation below T_c. Based on this idea of preformed pairs, a phenomenological field-theoretic model of high temperature superconductivity was proposed by Friedberg and Lee [91, 92]. The model is expressed by the following Lagrangian

$$\begin{aligned}\mathcal{L} =&\mathcal{L}_A + \mathcal{L}_\phi + \mathcal{L}_e + \mathcal{L}_I, \quad where \\ &\mathcal{L}_A = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \\ &\mathcal{L}_\phi = -|\left(\partial_\mu - i2eA_\mu\phi)\right)|^2 - M^2\phi^\dagger\phi, \\ &\mathcal{L}_e = \psi^\dagger\left(i\partial_t - eA_0 - m\right)\psi - \frac{1}{2m}|\left(\vec{\nabla} + ie\vec{A}\right)\psi|^2, \\ &\mathcal{L}_I = g(2M)^{1/2}\left(\phi^\dagger\psi_\uparrow\psi_\downarrow + \phi\psi_\downarrow^\dagger\psi_\uparrow^\dagger\right).\end{aligned} \tag{2.1}$$

In this field theory there are localized pairs, described by a bosonic field ϕ of charge $2e$ and mass $M \sim 2m_e$, where m_e is the mass of the electron. These bosons are unstable and decompose into pairs of electrons with opposite spins and these electrons recombine to form bosons. Thus in a large system there is always a macroscopic distribution of bosons coexisting with fermions following their respective statistical distribution rules. At temperature below T_c these bosons condense, i.e. there is a large number of bosons in the zero momentum state which coexist with fermions. This type of system with bosons and fermions coexisting in thermal equilibrium is generically referred to as boson-fermion (B-F) model. It was shown by the authors that below a critical temperature T_C the long range order in such a system can always be described by zero momentum bosonic amplitude B of the bosonic field. Also the the gap energy of the fermionic system is related to B by $\Delta^2 = |gB|^2$, where g is the coupling constant for the coupling of bosonic and fermionic fields. It is argued that as the transition from the normal phase to superconducting phase happens through a bose condensation of charge 2e bosons (which is of statistical origin), the transition temperature T_c can be higher.

Appearance of pseudogap in some particular region of Fermi surface lead some authors to postulate about a boson-fermion model for underdoped cuprates which is qualilatively different from that discussed previously [93]. They assumed that this gap formation above T_C is due to pairing of electrons in the cornoers of the Fermi surface into bosons which later condense to form Bose-Einstein condensate. But only the condensation of bosonic pairs would make the transition look like a BEC transition which has wide fluctuations near T_C, while the experimental evidence says that the transition is more mean field like with a small order parameter near T_C. To overcome this discrepancy authors had assumed that the bose condensation of bosonic pairs happens in the background of a fermi liquid. This assumption reconciles between the BEC picture and mean field like nature of transition. Apart from these two earlier works several other works along the same line of research exists [94–97].

This scenario of preformed pairs(and hence the boson-fermion model) is also discussed in another broad class of theories called "BCS- Bose-Einstein condensate(BEC) crossover" [98, 99]. This picture of "BCS-BEC" cross over has been investigated for many years as a possible explanation of "pseudogap" in the normal state of cuprate superconductors. In the cold-atom systems this cross over scenario has been realized though experiments by varying the strength of interaction by applying magnetic field in a controlled way (via Feshbach resonances). In such scenarios one starts with a weakly interacting theory of fermions which gives rise to BCS theory of superconductivity or theory of superfluidity. The interaction strength is then varied and made arbitrarily strong. In the strong coupling limit one would have tightly bound pairs of fermions which may act like bosonic molecules (Cooper pairs). The condensation of such bosonic Cooper pairs then gives rise to BEC. Thus there is a continuous evaluation from BCS in the weak coupling limit to BEC in the strong coupling limit. In the intermediate zone the system is a mixture of fermions and metastable pairs. In this way the boson-fermion model connects to this "BCS-BEC" picture.

There is another slightly different appearance of this model found in the description of different phases and phase transition of some systems of contemporary condensed matter physics and can be expressed by the following action written in imaginary time formalism as

$$\begin{aligned}\mathcal{S} = \int d^d x d\tau \Big(\mathcal{L}_A + \mathcal{L}_\phi + \mathcal{L}_e + \mathcal{L}_I\Big), \quad & where \\ \mathcal{L}_A = \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, & \\ \mathcal{L}_\phi = |(\partial_0 - i2eA_0\phi))|^2 + v_B \left|\Big(\partial_i - i2\frac{e}{c} A_i\phi)\Big)\right|^2 - M^2 \phi^\dagger \phi - \frac{\lambda}{2}|\phi|^4, & \\ \mathcal{L}_e = \bar{\psi}\left[\gamma_0 \left(i\partial_0 - ie\gamma_5 A_0\right)\psi + v_F \gamma_i \left(\partial_i + i\frac{e}{c}\gamma_5 A_i\right)\psi\right], & \\ \mathcal{L}_I = g\left(Re(\phi)\bar{\psi}\psi + Im(\phi)\bar{\psi} i\gamma_5 \psi\right). &\end{aligned} \tag{2.2}$$

This action is applicable to the context of semimetal-superconductor quantum phase transition for graphene and surface states of topological insulators [100], transition into the Kekul'e valence bond insulator in graphene and surface-hybridizing and time-reversal symmetric insulator in thin topological insulators [101], emergent space-time supersymmetry at disordered quantum critical points [102] etc.

Apart from those above mentioned contexts mixtures of bosons and fermions are studied in many other contexts as well, both experimental and theoretical. Experimental work on the properties of a mixture of Bose and Fermi gases include study of quantum degeneracy [103–106] and interactions [107,108]. Boson-fermion mixtures of dilute atomic and molecular gases at low temperatures are also studied theoretically in optical lattices to study their quantum phases including superfluid-insulator transition [109–111]. Boson-fermion systems also appear in studies of superconductor-insulator transitions [112, 113], of charged Bose liquids [114], etc.

2.1.2 Overview of our work:

The boson-fermion model of Friedberg and Lee closely resembles the Abelian Higgs model [115–118] as a field theory, including the appearance of vortices [92]. These vortices carry quantized magnetic flux, as can be derived from the minimum energy condition. The Abelian Higgs model in 3+1 dimensions contains vortex lines [119] which are minimum energy solutions of the field equations with topologically nontrivial boundary conditions. The interaction between these lines, or strings, is mediated by a 2-form (2-index antisymmetric tensor) gauge potential $B_{\mu\nu}$ called the Kalb-Ramond field [120]. In the symmetry broken phase and when vortex strings are present, the Abelian Higgs model can be written in terms of the string world sheet and the 2-form gauge field $B_{\mu\nu}$ using dualization [121–125]. Our goal for this work is to do this with fermions, i.e., to dualize the boson-fermion system in presence of vortices and reach what should be a useful starting point for the interaction of vortex lines with unpaired charged fermions.

Of particular interest is the coupling between the 2-form gauge field $B_{\mu\nu}$ and fermions. In an earlier work it was proposed that the 2-form field couples nonlocally to a topologically conserved current of the electrons [59],

$$\int d^4x B_{\mu\nu}\frac{1}{\Box}J^{\mu\nu}, \tag{2.3}$$

where $J^{\mu\nu} = \epsilon^{\mu\nu\rho\lambda}\partial_\rho(\bar{\psi}\gamma_\lambda\psi)$ is the 4-dimensional "curl" of the conserved fermion current. The nonlocal current $\frac{1}{\Box}J^{\mu\nu}$, more specifically its $\{0i\}$ component, contains the spin magnetic moment density as a contribution from the spin part. Then we can say that the coupling $B_{0i}\frac{1}{\Box}J^{0i}$ corresponds to a spin-spin interaction mediated via the 2-form gauge field B. If we find this interaction here, we will be able to say that we have found a local theory, namely that of the boson-fermion mixture, which has a description containing this "spin-gauge interaction". The electrons interact via photons as well, and quantum corrections due to fermion loops give rise to an effective $B\wedge F$ interaction. This term is central to the topological mass mechanism in 3+1 dimensions [57], analogous to the Chern-Simons term in 2+1 dimensions [126–128] which can also be generated by fermion loops [129]. As we shall see that the induced $B\wedge F$ term, in presence of massive $B_{\mu\nu}$ field leads to modification of the inter electron interaction in such a way that at large distance only a Coulomb term will contribute. We compare this with the destruction of Meissner effect by defect condensation as shown by Kleinert [64].

It is known for quite some time that a system containing particles and string like excitations may provide relalization of bosonic topological insulators (BTI) and the corresponding topological field theory contains $B\wedge F$ term in their action [54, 130]. As our system contains vortex lines and fermions and their interactions and a $B\wedge F$ term is induced due to quantum effects, it may also be a possible candidate for BTI. But we have not investigated this direction any further here.

The outline of this chapter is as follows. In Sec. 2.2 we dualize the Abelian Higgs model in the presence of vortex lines (strings) and charged fermions which couple through electromagnetic interactions, culminating in a non-local dual Lagrangian involving strings and the 2-form field which mediates interstring interactions. In Sec. 2.3 we derive the effective action by taking into account 1-loop corrections due to fermion loops. This generates a $B\wedge F$ interaction, which affects the propagators of both $B_{\mu\nu}$ and A_μ. Then in Sec. 2.4 we calculate the static potential between nonrelativistic fermions taking into account all the interactions as well as the 1-loop correction and end with some comments.

2.2 Dual Lagrangian for Boson-Fermion system

We first determine the dual of the field theory describing a boson-fermion system in the presence of vortices. Even though all particles in this system move non-relativistically, we will work with a four dimensional relativistic field theory. This is because the field theoretic duality we consider is most conveniently constructed in four dimensions and for relativistic theories and also because we will be able to use several standard results from usual quantum electrodynamics.

We start with the Abelian Higgs model where the gauge field A_μ is minimally coupled to unpaired charged fermions in addition to the complex scalar Higgs field. The Lagrangian of our system is thus

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D^\mu\phi^\dagger D_\mu\phi + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - V(\phi^\dagger\phi) - eA_\mu\bar{\psi}\gamma^\mu\psi\,, \tag{2.4}$$

where ϕ is a complex scalar field of charge q ($2e$ if ϕ describes Cooper pairs), ψ is the fermionic field with charge e, $D_\mu\phi = \partial_\mu + iqA_\mu\phi$ and $V(\phi^\dagger\phi) = \dfrac{\lambda}{4}(\phi^\dagger\phi - v^2)^2$ is the symmetry breaking potential.

As is well known, topologically stable structures like vortices or flux tubes can appear in this theory because the circle on which ϕ lies at its minimum is mapped on a circle at infinity. If there is a vortex in a plane, corresponding to a flux tube cutting through the plane as we will consider, the phase of ϕ becomes multivalued as we go around a circle at infinity. The vacuum condition $D_\mu\phi = 0$ then leads to quantization of magnetic flux in the vortex,

$$\oint_C A_\mu dx^\mu = -\frac{2n\pi}{q}\,, \tag{2.5}$$

where C is a circle at infinity and n is the winding number, i.e., the number of times the phase of ϕ winds around the vortex. It is a topological quantum number and describes the quantization of topological charge, while $\dfrac{2\pi}{q}$ is the quantum of magnetic flux passing through the vortex. To consider vortices explicitly we express ϕ in polar form,

$$\phi = vf\exp(i\chi)\,. \tag{2.6}$$

The function f vanishes along the core of the flux tube and reaches $f = 1$ far from the core region[1]. The Lagrangian, including a gauge-fixing term, then takes the form

$$\begin{aligned}\mathscr{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}v^2\partial_\mu f\partial^\mu f + \frac{1}{2}v^2f^2(\partial_\mu\chi + qA_\mu)(\partial^\mu\chi + qA^\mu)\\ &- \frac{1}{2\xi}(\partial_\mu A^\mu)^2 - \frac{\lambda}{4}(f^2-1)^2 + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - eA_\mu\bar{\psi}\gamma^\mu\psi\,.\end{aligned} \tag{2.7}$$

We shall dualize the theory, starting from the partition function

$$\mathcal{Z} = \int \mathscr{D}A_\mu\mathscr{D}f\mathscr{D}\chi\mathscr{D}\bar{\psi}\mathscr{D}\psi\exp\left(i\int d^4x\,\mathscr{L}\right)\,. \tag{2.8}$$

The duality transformation takes us from the Higgs picture above, where the degrees of freedom are adequately described by a charged Higgs minimally coupled to the electromagnetic gauge field, to an equivalent vortex picture in which vortices interact through

[1] The zero of f on a plane is the location of a vortex. A locus of zeroes in space defines a vortex string or a flux tube.

the second rank antisymmetric tensor Kalb-Ramond field. To implement this we first linearize the term $\frac{1}{2}v^2 f^2(\partial_\mu \chi + qA_\mu)^2$ by introducing an auxiliary field through a Gaussian integral into the partition function as

$$N \int \mathscr{D}C_\mu \exp\left\{ \left(-i \int d^4x \left[\frac{C_\mu}{\sqrt{2}v} + \frac{v}{\sqrt{2}} f(\partial_\mu \chi + qA_\mu) \right]^2 \right) \right\} = 1\,. \tag{2.9}$$

Then we can write the partition function as

$$\begin{aligned} \mathcal{Z} = \int & \mathscr{D}A_\mu \mathscr{D}f \mathscr{D}\chi \mathscr{D}\bar{\psi} \mathscr{D}\psi \mathscr{D}C_\mu \\ & \exp\left(i \int d^4x \Big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}v^2 \partial_\mu f \partial^\mu f - \frac{C_\mu C^\mu}{2v^2} - C^\mu f(\partial_\mu \chi + qA_\mu) \right. \\ & \left. - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 - V(f^2) + \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - eA_\mu \bar{\psi}\gamma^\mu \psi \Big) \right). \end{aligned} \tag{2.10}$$

As mentioned earlier, the phase χ is multivalued around a vortex string and the value of χ changes by $2n\pi$ as one goes around the vortex string n times, n being the winding number. So we decompose $\chi = \chi^r + \chi^s$, where the superscript s indicates the singular part of χ describing a vortex configuration and r denotes the regular part which is single valued and corresponds to fluctuations around a given vortex configuration. By doing an integration by parts on the term $C^\mu f \partial_\mu \chi^r$ we can shift the partial derivative onto $C^\mu f$ and then integrate over χ^r producing a delta function. Thus we can write

$$\begin{aligned} \mathcal{Z} = \int & \mathscr{D}A_\mu \mathscr{D}f \mathscr{D}\chi^s \mathscr{D}\bar{\psi} \mathscr{D}\psi \mathscr{D}C_\mu \delta(\partial_\mu(C^\mu f)) \\ & \exp\left(i \int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}v^2 \partial_\mu f \partial^\mu f - \frac{C_\mu C^\mu}{2v^2} - C^\mu f(\partial_\mu \chi^s + qA_\mu) \right.\right. \\ & \left.\left. - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 - V(f^2) + \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - eA_\mu \bar{\psi}\gamma^\mu \psi \right) \right). \end{aligned} \tag{2.11}$$

The delta function can be solved by introducing an antisymmetric tensor (2-form) potential $B_{\mu\nu}$ and setting $C^\mu = \frac{1}{2f}\varepsilon^{\mu\nu\rho\lambda}\partial_\nu B_{\rho\lambda}$, which allows us to write the partition function as

$$\begin{aligned} \mathcal{Z} = \int & \mathscr{D}A_\mu \mathscr{D}f \mathscr{D}\chi^s \mathscr{D}\bar{\psi} \mathscr{D}\psi \mathscr{D}B_{\mu\nu} \\ & \exp\left(i \int d^4x \Big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}v^2 \partial_\mu f \partial^\mu f + \frac{1}{12v^2 f^2} H^{\nu\rho\lambda} H_{\nu\rho\lambda} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \right. \\ & \left. - \frac{1}{2}\varepsilon^{\mu\nu\rho\lambda} B_{\rho\lambda}\partial_\mu \partial_\nu \chi^s - \frac{q}{2}\varepsilon^{\mu\nu\rho\lambda}\partial_\nu B_{\rho\lambda} A_\mu - V(f^2) + \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - eA_\mu \bar{\psi}\gamma^\mu \psi \Big) \right). \end{aligned} \tag{2.12}$$

Here we have defined $H_{\nu\rho\lambda} = \partial_\nu B_{\rho\lambda} + \partial_\rho B_{\lambda\nu} + \partial_\lambda B_{\nu\rho}$ as the field strength of the 2-form field. The curl of the velocity of the scalar field (or supercurrent) is called vorticity, $\Sigma^{\rho\lambda} = \varepsilon^{\mu\nu\rho\lambda}\partial_\mu \partial_\nu \chi^s$. Around a vortex this quantity is non-zero and in case of a straight rod like array of vortices, i.e. a vortex line or a flux tube, along the Z-axis, it is given by

$$(\partial_x \partial_y - \partial_y \partial_x)\chi^s = 2n\pi \delta^2(\vec{\rho}). \tag{2.13}$$

This expresses the location of the vortices in the $X-Y$ plane. By dualizing it we get the world sheet of the vortex line in 3+1 dimensions,

$$\begin{aligned}\Sigma^{\rho\lambda} &= \varepsilon^{\mu\nu\rho\lambda}\partial_\mu\partial_\nu\chi^s \\ &= \int d\sigma_{\mu\nu}\delta(x-X)\,. \end{aligned} \tag{2.14}$$

X^μ are the coordinates of the world sheet of the vortex line and $d\sigma_{\mu\nu} = d\tau ds\dfrac{\partial(X_\mu, X_\nu)}{\partial(s,\tau)}$ is the surface element over the world sheet. Thus we write the partition function as

$$\begin{aligned}\mathscr{Z} = \int &\mathscr{D}A_\mu\mathscr{D}f\mathscr{D}\chi^s\mathscr{D}\bar{\psi}\mathscr{D}\psi\mathscr{D}B_{\mu\nu} \\ &\exp\left(i\int d^4x\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}v^2\partial_\mu f\partial^\mu f + \frac{1}{12v^2f^2}H^{\nu\rho\lambda}H_{\nu\rho\lambda} - \frac{1}{2}B_{\rho\lambda}\Sigma^{\rho\lambda}\right.\right. \\ &\left.\left.\qquad -\frac{q}{2}\varepsilon^{\mu\nu\rho\lambda}\partial_\nu B_{\rho\lambda}A_\mu - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 - V(f^2) + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - eA_\mu\bar{\psi}\gamma^\mu\psi\right)\right)\,. \end{aligned} \tag{2.15}$$

Next we shall integrate out A_μ from the theory to remove the off diagonal term $\varepsilon^{\mu\nu\rho\lambda}\partial_\nu B_{\rho\lambda}A_\mu$ and write the theory entirely in terms of $B_{\mu\nu}$ gauge field. For that we rename the currents $\frac{q}{2}\varepsilon^{\mu\nu\rho\lambda}\partial_\nu B_{\rho\lambda} = J^\mu_H$ and $e\bar{\psi}\gamma^\mu\psi = J^\mu_\psi$, separate out the terms which depend on A_μ from the rest of the partition function and then integrate over A_μ ,

$$\begin{aligned}\int \mathscr{D}A_\mu \exp\left(i\int d^4x\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + A_\mu(J^\mu_H + J^\mu_\psi)\right)\right) \\ = \mathscr{N}\exp\left(-\frac{i}{2}\int d^4xd^4y(J^\mu_H + J^\mu_\psi)\Delta_{\mu\nu}(J^\nu_H + J^\nu_\psi)\right)\,. \end{aligned} \tag{2.16}$$

Here $\Delta_{\mu\nu}$ is the Green's function corresponding to the operator

$$\Delta^{-1}_{\mu\nu} = g^{\mu\nu}\Box - (1-\frac{1}{\xi})\partial^\mu\partial^\nu\,. \tag{2.17}$$

In momentum space it is given by

$$\Delta_{\mu\nu}(k) = -\frac{g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}}{k^2 + i\epsilon}\,. \tag{2.18}$$

We will suppress the $+i\epsilon$ in what follows, but it is present in each of the propagators appearing below. The integration over A_μ has produced a normalization factor $\mathscr{N}$ which does not contribute to the rest of the partition function. We can thus write the action as

$$\begin{aligned}S = \int d^4x&\left(\frac{1}{2}v^2\partial_\mu f\partial^\mu f + \frac{1}{12v^2f^2}H^{\nu\rho\lambda}H_{\nu\rho\lambda} - \frac{1}{2}B_{\rho\lambda}\Sigma^{\rho\lambda} - V(f^2) + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi\right) \\ &- \frac{1}{2}\int d^4xd^4y\left(J^\mu_H + J^\mu_\psi\right)(x)\Delta_{\mu\nu}(x,y)\left(J^\nu_H + J^\nu_\psi\right)(y)\,. \end{aligned} \tag{2.19}$$

We can further simplify the last term. Note that since J^μ_H is a (topologically) conserved current, the second term in $\Delta_{\mu\nu}$ annihilates it, so we can write

$$\frac{1}{2}\int d^4xd^4yJ^\mu_H(x)\Delta_{\mu\nu}(x,y)J^\nu_H(y) = -\int d^4x\frac{q^2}{12}H_{\nu\rho\lambda}\frac{1}{\Box}H^{\nu\rho\lambda}\,, \tag{2.20}$$

as well as

$$\int d^4x d^4y\ J^\mu_H(x)\Delta_{\mu\nu}(x,y)J^\nu_\psi(y) = \int d^4x\ \frac{1}{2}eqB^{\mu\nu}\varepsilon_{\mu\nu\rho\lambda}\partial^\rho\frac{1}{\Box}\bar{\psi}\gamma^\lambda\psi\,. \tag{2.21}$$

Above we have used the notation $(\Box)^{-1}$ which actually means the Green's function for the $(\Box)$ operator. In order to understand the remaining part, which is quadratic in the fermion current J^μ_ψ, we note that it is exactly what we would get if we integrate over A_μ ordinary quantum electrodynamics, i.e.

$$\begin{aligned}\int \mathscr{D}A_\mu \exp\left(i\int d^4x\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu J^\mu_\psi + \frac{1}{2\xi}(\partial_\mu A^\mu)^2\right)\right)\\ = \mathscr{N}_0 \exp\left(-\frac{i}{2}\int d^4x d^4y\ J^\mu_\psi(x)\Delta_{\mu\nu}(x,y)J^\nu_\psi(y)\right)\,,\end{aligned} \tag{2.22}$$

where $\mathscr{N}_0$ is a normalization factor.

Thus after collecting all these terms, we can write the dual Lagrangian as

$$\boxed{\begin{aligned}\mathscr{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - eA_\mu\bar{\psi}\gamma^\mu\psi - \frac{1}{2}eqB^{\mu\nu}\varepsilon_{\mu\nu\rho\lambda}\partial^\rho\frac{1}{\Box}\bar{\psi}\gamma^\lambda\psi + \frac{1}{2}v^2\partial_\mu f\partial^\mu f\\ &+\frac{1}{12v^2}H^{\nu\rho\lambda}\left(\frac{1}{f^2}+\frac{q^2v^2}{\Box}\right)H_{\nu\rho\lambda} - \frac{1}{2}B_{\rho\lambda}\Sigma^{\rho\lambda} - V(f^2)\,,\end{aligned}} \tag{2.23}$$

where we have suppressed gauge-fixing terms for A_μ or $B_{\mu\nu}$. Thus starting from a system containing vortex strings and described by an Abelian Higgs model in the broken phase in which charged fermions are also present, we have arrived at the dual Lagrangian of Eq. (2.23) in which the Kalb-Ramond field $B_{\mu\nu}$ [120] couples to a topologically conserved non-local tensor current,

$$J^{\mu\nu} = \frac{1}{2}q\varepsilon^{\mu\nu\rho\lambda}\Box^{-1}\partial_\rho J_\lambda\,, \tag{2.24}$$

with J^μ being the conserved electron current. The conserved charge density J^{0i} for this current can be split into orbital and spin parts, with the spin contribution for nonrelativistic electrons being the intrinsic spin density of the electron,

$$\left(J^{0i}_{\text{spin}}\right)_{\text{NR}} \propto \psi^\dagger\sigma^i\psi\,, \tag{2.25}$$

up to dimensionful constants, when the charge is time-independent and cannot accumulate. Thus in other words we have a gauge theory in which the gauge potential mediating string-string interaction couples to the spin current of charged fermions [59].

The Lagrangian of Eq. (2.23) is invariant, not only with respect to the usual gauge transformation $A_\mu \to A_\mu + \partial_\mu\lambda$ for arbitrary real functions λ, but also under the vector (or extended or higher or Kalb-Ramond) gauge transformation $B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu\Lambda_\nu - \partial_\mu\Lambda_\nu$, provided

$$\partial_\mu\Sigma^{\mu\nu} = 0\,. \tag{2.26}$$

This shows that vortex lines must either form closed loops or be infinitely long (or end at the boundaries of the superconducting region) as the world-sheet current is conserved by itself. This is of course expected as magnetic field lines must either close on themselves or go out to infinity, since there are no magnetic monopoles.

2.3 Induced $B \wedge F$ Term

In order to see the effect of the nonlocal coupling on the boson-fermion system, let us calculate the quantum corrections at one fermion loop. We will do this by first setting $f \to 1$, which corresponds to the limit of the flux tubes being very thin. We also redefine $\frac{1}{v}B_{\mu\nu}$ as $B_{\mu\nu}$ for convenience of calculations.

The partition function then becomes

$$\begin{aligned}\mathcal{Z} = \int &\mathcal{D}\chi^s \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}B_{\mu\nu}\mathcal{D}A_\mu \\ &\exp\left(i\int d^4x\Big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - eA_\mu\bar{\psi}\gamma^\mu\psi \right. \\ &\left. - \frac{1}{2}eMB^{\mu\nu}\varepsilon_{\mu\nu\rho\lambda}\partial^\rho\frac{1}{\Box}\bar{\psi}\gamma^\lambda\psi + \frac{1}{12}H_{\nu\rho\lambda}(1+M^2\,\Box^{-1})H^{\nu\rho\lambda} - \frac{v}{2}B_{\rho\lambda}\Sigma^{\rho\lambda}\Big)\right), \end{aligned} \tag{2.27}$$

where we have written $M = qv$. The nonlocal term involving $M^2\,\Box^{-1}$ is a mass term for $B_{\mu\nu}$ and is sometimes called a Meissner term for that reason [131, 132]. The nonlocal interaction term between the $B_{\mu\nu}$-field and the fermion can be written as

$$\int d^4x\,\frac{1}{2}eMB^{\mu\nu}\varepsilon_{\mu\nu\rho\lambda}\partial^\rho\frac{1}{\Box}\bar{\psi}\gamma^\lambda\psi = \int d^4x\,\frac{1}{2}eM\varepsilon_{\mu\nu\rho\lambda}\bar{\psi}\gamma^\mu\psi\frac{1}{\Box}\partial^\nu B^{\rho\lambda}\,, \tag{2.28}$$

as can be seen directly from where it first appeared in Eq. (2.21). For convenience of calculations let us now define an "effective gauge field" A^{eff}_μ as $A^{\text{eff}}_\mu = A_\mu + M\Box^{-1}F_\mu$ where we have written $F_\mu = \frac{1}{2}\varepsilon_{\mu\nu\rho\lambda}\partial^\nu B^{\rho\lambda}$. Quantum corrections to the action due to fermion loops are calculated in the standard textbook method [133, 134]: We expand $\psi = \psi_0 + \eta$, where ψ_0 is a solution of the equation of motion $\frac{\delta S}{\delta\bar{\psi}}\Big|_{\psi=\psi_0} = 0$, similarly for $\bar{\psi}_0$, then integrate over $\eta, \bar{\eta}$ to first order in e^2 for one loop,

$$\begin{aligned}\int \mathcal{D}\bar{\eta}\mathcal{D}\eta\,\exp&\left(i\int d^4x\,\bar{\eta}\left(i\gamma^\mu\partial_\mu - m - e\gamma^\mu A^{\text{eff}}_\mu\right)\eta\right) \\ &\sim \exp\left[-\frac{i}{2}\int\frac{d^4k}{(2\pi)^4}\Pi(k^2)A^{\text{eff}}_\mu(-k)\left(g^{\mu\nu}k^2 - k^\mu k^\nu\right)A^{\text{eff}}_\nu(k)\right]. \end{aligned} \tag{2.29}$$

Here A^{eff}_μ is defined as before, while $\Pi(p^2)$ includes the effect of modes up to a cutoff Λ and is given by[2].

$$\Pi(k^2) = \frac{e^2}{2\pi^2}\int_0^1 z(1-z)\left[\ln\left(1 + \frac{\Lambda^2}{m^2 - k^2z(1-z)}\right) - \frac{\Lambda^2}{\Lambda^2 + m^2 - k^2z(1-z)}\right], \tag{2.30}$$

for a cutoff Λ. The natural cutoff scale in the system of vortices and electrons is the thickness of the vortex string ($\xi_\lambda \sim 1/\sqrt{\lambda}v$). This is typically comparable to the atomic scale, so we set $\Lambda \ll m$[3] (also $|k^2| \ll m^2$) to find at the leading order

$$\Pi(k^2) = \frac{e^2}{24\pi^2}\frac{\Lambda^4}{m^4} + \cdots\,, \tag{2.31}$$

[2] Details of this calculation is given in Appendics-.2.

[3] This approximation may not be true for quasi electrons which can have very small mass as well. See Sec. 2.5 for an elaborate discussions.

ignoring terms of the order of $\frac{\Lambda^6}{m^6}, \frac{\Lambda^4 k^2}{m^2}, \frac{\Lambda^2 k^4}{m^6}$. We can now write the Lagrangian including the loop correction as

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{12}H_{\nu\rho\lambda}(1 + M^2\,\Box^{-1})H^{\nu\rho\lambda} - \frac{v}{2}B_{\rho\lambda}\Sigma^{\rho\lambda} - \frac{e^2}{24\pi^2}\frac{\Lambda^4}{m^4}\left(\frac{1}{4}F^{\text{eff}}_{\mu\nu}F^{\text{eff}\mu\nu}\right), \tag{2.32}$$

where we have suppressed terms involving ψ_0. Recalling the definition of A^{eff}_μ, we can write

$$F^{\text{eff}}_{\mu\nu}F^{\text{eff}\mu\nu} = F_{\mu\nu}F^{\mu\nu} - M\epsilon_{\mu\nu\rho\lambda}F^{\mu\nu}B^{\rho\lambda} + \frac{1}{3}M^2H_{\mu\nu\lambda}\frac{1}{\Box}H^{\mu\nu\lambda}, \tag{2.33}$$

after taking into account an integration by parts. Writing $Z = \frac{e^2}{24\pi^2}\frac{\Lambda^4}{m^4}$, we rewrite the Lagrangian as

$$\begin{aligned}\mathscr{L} = &-\frac{1}{4}\left(1+Z\right)F_{\mu\nu}F^{\mu\nu} + \frac{1}{12}H^{\nu\rho\lambda}\left(1 + (1-Z)\frac{M^2}{\Box}\right)H_{\nu\rho\lambda} + \frac{1}{4}ZM\epsilon_{\mu\nu\rho\lambda}F^{\mu\nu}B^{\rho\lambda}\\ &- \frac{v}{2}B_{\rho\lambda}\Sigma^{\rho\lambda} + \bar\psi_0(i\gamma^\mu\partial_\mu - m)\psi_0 - eA_\mu\bar\psi_0\gamma^\mu\psi_0 - \frac{1}{2}eMB^{\mu\nu}\varepsilon_{\mu\nu\rho\lambda}\partial^\rho\frac{1}{\Box}\bar\psi_0\gamma^\lambda\psi_0\,.\end{aligned} \tag{2.34}$$

If we now rescale $A_\mu \to \sqrt{1+Z}A_\mu$ and also define the "renormalized charge" $e_R^2 = e^2\left(1+Z\right)^{-1} \simeq e^2(1-Z)$, we obtain the Lagrangian in the form

$$\begin{aligned}\mathscr{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{12}H^{\nu\rho\lambda}\left(1 + \frac{M_R^2}{\Box}\right)H_{\nu\rho\lambda} + \frac{1}{4}ZM_R\epsilon_{\mu\nu\rho\lambda}F^{\mu\nu}B^{\rho\lambda} - \frac{v}{2}B_{\rho\lambda}\Sigma^{\rho\lambda}\\ &+ \bar\psi_0(i\gamma^\mu\partial_\mu - m)\psi_0 - e_RA_\mu\bar\psi_0\gamma^\mu\psi_0 - \frac{1}{2}e_RM_RB^{\mu\nu}\varepsilon_{\mu\nu\rho\lambda}\partial^\rho\frac{1}{\Box}\bar\psi_0\gamma^\lambda\psi_0\,.\end{aligned} \tag{2.35}$$

Here we have written $M_R^2 = M^2\left(1-Z\right) \simeq \frac{q^2v^2}{(1+Z)}$ as the "renormalized mass" of the gauge boson. Note that we can write $M_R = q_Rv$ since all electric charges should be renormalized the same way, so that $q_R = \frac{q}{\sqrt{1+Z}}$. There is also an induced $B\wedge F$ term with a coefficient which depends on the cutoff Λ.

The coefficient of the induced $B\wedge F$ term is very small and depends on the cutoff, which is in turn determined by the properties of the system. As we shall see now, this term will increase the mass of the gauge fields A_μ or $B_{\mu\nu}$. By the mass of the gauge fields we mean the pole of the propagator of the relevant field, which can be calculated either by summing an infinite series or by taking only the part of the Lagrangian quadratic in the fields and eliminating one gauge field in favor of the other. Let us use the second method to find the poles in the propagators, starting with $B_{\mu\nu}$. We separate out the quadratic terms containing A_μ,

$$\mathscr{Z}_A = \int\mathscr{D}A_\mu\exp\left(i\int d^4x\left(\frac{1}{2}A_\mu K^{\mu\nu}A_\nu + \frac{1}{4}ZM_R\epsilon_{\mu\nu\rho\lambda}F^{\mu\nu}B^{\rho\lambda}\right)\right), \tag{2.36}$$

where $K_{\mu\nu}$ is the invertible operator $g_{\mu\nu}\Box - (1-\frac{1}{\xi})\partial_\mu\partial_\nu$. We complete the square and write

$$\mathscr{Z}_{\mathcal{A}} = \left(\int\mathscr{D}A'_\mu\exp i\int d^4x\frac{1}{2}A'_\mu K^{\mu\nu}A'_\nu\right)\exp\left(i\int d^4x\left(\frac{Z^2M_R^2}{12}H^{\sigma\rho\lambda}\frac{1}{\Box}H_{\sigma\rho\lambda}\right)\right). \tag{2.37}$$

The integration over A'_μ provides a normalization factor while the second term gets added to the Lagrangian. The mass term of $B_{\mu\nu}$ becomes $\frac{1}{12}(1+Z^2)M_R^2 H^{\nu\rho\lambda}\,\Box^{-1}H_{\nu\rho\lambda}$, resulting in a shift of the coefficient of the Meissner term so that $M_B = M_R\sqrt{1+Z^2}$ is the new mass of B . Thus the mass of the interstring gauge potential $B_{\mu\nu}$ increases because of quantum effects due to fermion loops.

Let us now see what happens to the propagator of A_μ if we integrate out $B_{\mu\nu}$ instead. We start from

$$\mathscr{Z}_B = \int \mathscr{D}B_{\mu\nu} \exp\left(-i\int d^4x \left(\frac{1}{4}B_{\mu\nu}M^{\mu\nu\rho\lambda}B_{\rho\lambda} - \frac{1}{4}ZM_R\epsilon^{\mu\nu\rho\lambda}B_{\mu\nu}F_{\rho\lambda}\right)\right), \tag{2.38}$$

where for convenience we have written

$$M^{\mu\nu\rho\lambda} = \left(\Box + M_R^2\right) g^{\mu[\rho}g^{\lambda]\nu} + \left(1 + M_R^2\Box^{-1} - \frac{1}{\eta}\right)\left(g^{\nu[\rho}g^{\lambda]\sigma}\partial_\sigma\partial^\mu - g^{\mu[\rho}g^{\lambda]\sigma}\partial_\sigma\partial^\nu\right) . \tag{2.39}$$

We can now perform the integral by completing the square and find

$$\mathscr{Z}_B = N' \exp\left(-\frac{i}{4}\int d^4x d^4y\, Z^2 M_R^2 F_{\mu\nu}\frac{1}{\Box + M_R^2}F^{\mu\nu}\right) . \tag{2.40}$$

This is added to the action for A_μ , so that the quadratic term in the Lagrangian becomes

$$-\frac{1}{4}F_{\mu\nu}\left(1 + \frac{Z^2M_R^2}{\Box + M_R^2}\right)F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 . \tag{2.41}$$

A similar expression was found in three dimensions in the closely related context of the disorder field [64] in a superconductor, but in the absence of charged fermions. The propagator of the field A_μ is

$$G_{\mu\nu} = -\left(\frac{g_{\mu\nu}}{p^2}\frac{p^2 - M_R^2}{p^2 - M_R^2(1+Z^2)} + \frac{p_\mu p_\nu}{p^4}\frac{p^2(\xi-1) - M_R^2\left(\xi(Z^2+1)-1\right)}{p^2 - M_R^2(1+Z^2)}\right) . \tag{2.42}$$

The second term in the Green's function will disappear when a conserved current couples to it, while the first term can be written as

$$G_{\mu\nu} = -\left(\frac{1}{1+Z^2}\frac{g_{\mu\nu}}{p^2} + \frac{Z^2}{1+Z^2}\frac{g_{\mu\nu}}{p^2 - M_R^2(1+Z^2)}\right) . \tag{2.43}$$

Unlike the usual mechanism of topological mass generation using a $B\wedge F$ interaction where the massive $B_{\mu\nu}$ field is 'dual' to the massive A_μ field in the sense that the propagating degrees of freedom can be described equally well by either field, the boson-fermion system shows a distinction between the two alternative descriptions in presence of vortices.

2.4 Force Law

Another window to the physics of the system is provided by the force law as experienced by the charged fermions, which we proceed to derive now. We will find it by integrating out both the gauge fields A_μ and $B_{\mu\nu}$ and then taking the nonrelativistic limit for the fermionic currents. We start by integrating out $B_{\mu\nu}$ from the Lagrangian of Eq. (2.35), which includes corrections up to one fermion loop,

$$\mathscr{Z}_B = \int \mathscr{D}B_{\mu\nu} \exp\left(-i\int d^4x \left(\frac{1}{4}B_{\mu\nu}M^{\mu\nu\rho\lambda}B_{\rho\lambda} - \frac{1}{2}B_{\mu\nu}J^{\mu\nu}\right)\right), \tag{2.44}$$

where we have written $J_{\mu\nu} = \dfrac{1}{2}ZM_R\epsilon_{\mu\nu\rho\lambda}F^{\rho\lambda} - v\Sigma_{\mu\nu} - e_RM_R\epsilon_{\mu\nu\rho\lambda}\partial^\rho(\Box^{-1})J^\lambda$, with the fermion current being $J^\lambda = \bar\psi_0\gamma^\lambda\psi_0$. Integrating over $B_{\mu\nu}$ we get

$$\begin{aligned}\mathscr{Z}_B \sim \exp\frac{i}{4}\int d^4xd^4y\,\Bigg(&-Z^2M_R^2F_{\mu\nu}\frac{1}{\Box+M_R^2}F^{\mu\nu} - 4Ze_RM_R^2A_\mu\frac{1}{(\Box+M_R^2)}J_\mu\\ &+2e_R^2M_R^2J_\lambda\frac{1}{\Box(\Box+M_R^2)}J^\lambda + v^2\Sigma^{\mu\nu}\frac{1}{\Box+M_R^2}\Sigma_{\mu\nu}\\ &-2vZM_R\varepsilon^{\mu\nu\rho\lambda}A_\mu\frac{1}{\Box+M_R^2}\partial_\nu\Sigma_{\rho\lambda} + 2ve_RM_R\varepsilon^{\mu\nu\rho\lambda}\Sigma_{\mu\nu}\frac{1}{\Box(\Box+M_R^2)}\partial_\rho J_\lambda\Bigg)\,.\end{aligned} \tag{2.45}$$

The first two terms and fifth term in the integral contribute to the action of A_μ , which is now integrated over to get the force law. The integral over A_μ now reads

$$\begin{aligned}\mathscr{Z}_A = \int \mathscr{D}A_\mu \exp i\int d^4x\,\Bigg[&-\frac{1}{4}F_{\mu\nu}\left(1+\frac{Z^2M_R^2}{\Box+M_R^2}\right)F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2\\ &-A_\mu\left(e_RJ^\mu + \frac{e_RZM_R^2}{\Box+M_R^2}J^\mu + vZM_R\frac{1}{\Box+M_R^2}K^\mu\right)\Bigg]\,,\end{aligned} \tag{2.46}$$

where we have written $K^\mu = \dfrac{1}{2}\varepsilon^{\mu\nu\rho\lambda}\partial_\nu\Sigma_{\rho\lambda}$. Integration over A_μ produces

$$\mathscr{Z}_A \sim \exp\left(-\frac{i}{2}\int d^4xd^4y\tilde J^\mu(x)G_{\mu\nu}(x-y)\tilde J^\nu(y)\right)\,, \tag{2.47}$$

where $G_{\mu\nu}$ is the Green's function calculated in Eq. (2.42) and $\tilde J^\mu = \left(1+\dfrac{ZM_R^2}{\Box+M_R^2}\right)e_RJ^\mu +$ $vZM_R\dfrac{1}{\Box+M_R^2}K^\mu$. To get the net interaction potential between fermions we now add the third term of Eq. (2.45) to the above integral and convert the result to momentum space. Finally we are left with the effective current-current interaction

$$\frac{e_R^2}{2}\int\frac{d^4p}{(2\pi)^4}J^\mu(-p)\left(\frac{(1-Z)^2}{1+Z^2}\frac{1}{p^2-M_R^2(1+Z^2)} + \frac{2Z}{1+Z^2}\frac{1}{p^2}\right)J_\mu(p)\,. \tag{2.48}$$

This represents, for the current of non-relativistic fermions, a Yukawa potential in the leading order along with a very small Coulomb correction. The expression (2.46) also includes the vortex-vortex and vortex-fermion interaction terms and combining them with the relevant terms in $\mathscr{Z}_B$ we get

$$-\frac{v^2}{4}\int\frac{d^4p}{(2\pi)^4}\Sigma_{\mu\nu}(-p)\frac{1}{p^2-M_R^2(1+Z^2)}\Sigma^{\mu\nu}(p)\,, \tag{2.49}$$

which gives the interaction between two vortex lines and

$$\frac{ive_R(1-Z)M_R}{2}\varepsilon^{\mu\nu\rho\lambda}\int\frac{d^4p}{(2\pi)^4}\Sigma_{\mu\nu}(-p)\frac{1}{p^2(p^2-M_R^2(1+Z^2))}p_\rho J_\lambda(p)\,, \tag{2.50}$$

which gives the vortex-fermion interaction. Since in the non-relativistic static limit we can write $\varepsilon^{0ijk}\partial_j \frac{1}{\Box} J_k \sim S^i$, the spin magnetic moment density of a static electron, the above expression gives an effective vortex-spin interaction.

2.5 Discussion

In this paper we analyzed the interaction of vortices in an Abelian Higgs model with charged fermions. This may be thought of as a field theoretic description of a type II superconductor with thin tubes of magnetic flux, in which unpaired electrons coexist with the charged pairs and interact electromagnetically through their minimal coupling with the photon. Dual formulation of the system using the four dimensional relativistic theory leads to a nonlocal interaction term between the antisymmetric tensor field and fermions, equivalent to a gauge field coupled to the spin density current of the fermions. This provides a post-facto justification of working with a relativistic formulation in four dimensions, for the spin of fermions appears naturally in it.

One motivation of this work was to see if the vortex-electron interaction could give rise, in the dual picture, to the nonlocal coupling of the two-form field with electrons proposed earlier in [59]. We found this, as an 'emergent' interaction involving the spin current of the electrons that does not appear in the original way of writing the model but emerges in the process of dualization. Often the dual picture of the Abelian Higgs model in the context of a type II superconductor is studied as a nonrelativistic field theory (often in two spatial dimensions), leading to the disorder field [64,135], analogous to the antisymmetric tensor potential. We note however that since spin has to be introduced by hand in a non-relativistic theory, this interaction with the spin current would not have emerged from the non-relativistic field theory calculations usually done for superconductors.

We have also found, as had been shown earlier, that this interaction generates a $B\wedge F$ term in one-loop effective action. This increases the mass of both gauge fields A_μ and $B_{\mu\nu}$, which should decrease the penetration depth. Here we have encountered a Coulomb term in the interaction potential between two charges, with a very high dielectric constant $\kappa \sim (2Z)^{-1} = \dfrac{e^2}{12\pi^2}\dfrac{\Lambda^4}{m^4}$. As we mentioned previously that the largest energy scale in the type-II superconductor is is the width of the vortex or the coherence length of Cooper pairs which is $\xi_\lambda \sim \sqrt{\lambda}v$. Thus one may replace Λ by this scale and get $\kappa = \dfrac{e^2}{12\pi^2}\dfrac{\lambda^2 v^4}{m^4}$. Thus we see that this effect appears purely due to presence of superconducting background as vanishes in the limit of vanishing vev $v \to 0$.

The quantum corrected effective theory obtained in our work can be compared with the theory of disorder condensation discussed in [64]. The disorder field theory of superconductors in 2+1 dimension proposed by H. Kleinert is given by

$$\beta H_{SC} = \int d^3x \left[\frac{q^2}{2\beta m_A^2}\left(\vec{\nabla}\times\vec{a}\right)^2 - iq\vec{a}\cdot\left(\nabla\times\vec{A}\right) + \frac{\beta}{2}\left(\nabla\times\vec{A}\right)^2 - i\vec{a}\cdot j^v + \frac{\beta\epsilon_c}{2} j^{v2}\right], \tag{2.51}$$

where $\vec{a}$ is the gauge field, mediating interaction between vortices which are represented by j^v, β is the equilibrium temperature, q is the charge of the condensate field, m_A is the mass of the electromagnetic gauge field $\vec{A}$. When tmperature is high ($\beta \to 0$) vortices become prolific and in such a situation one may integrate out J^v like an ordinary gauge theory and thus get a mass term of $\vec{a}$ gauge field.

$$\beta H_{SC} = \int d^3x \left[\frac{q^2}{2\beta m_A^2} \left[\left(\vec{\nabla} \times \vec{a} \right)^2 + m_a^2 \vec{a}_T^2 \right] - iq\vec{a} \cdot \left(\nabla \times \vec{A} \right) + \frac{\beta}{2} \left(\nabla \times \vec{A} \right)^2 \right] \tag{2.52}$$

If we now integrate out $\vec{a}_T$ from the theory the Hamiltonian becomes like

$$\beta H_{SC} = \frac{1}{2} \int d^3x \left(\nabla \times \vec{A} \right) \left(1 + \frac{m_A^2}{-\nabla^2 + m_a^2} \right) \left(\nabla \times \vec{A} \right) \tag{2.53}$$

The above equation can be compared with Eq. (2.41) which was obtained by calculating loop correction in our dual model. It was suggested by Kleinert that for the physics of large distance one can expand $\frac{1}{-\nabla^2 + m_a^2}$ in powers of ∇^2 and therefore can see that no mass term emerges and the interaction between static fermions become long range Coulomb like interaction. Although Kleinert's work concentrate on Euclidean 3 dimensional manifold, we see that a similar effective field theoretic description has also occured in our case just from fermionic loop corrections in the dual description. Due to this the inter electron interaction potential is modified by a Coulombic term which survives at large distance.

Also the factor $\kappa^{-1} = \frac{e^2}{12\pi^2} \frac{\Lambda^4}{m^4}$, which seems to act as an inverse effective dielectric contant, may not be very small. This is because effective charge and mass of electrons in material system are highly dependent on the band properties and therefore varies from system to system. As an example, If we try to model the interface of superconductor-topological insulator hybrid by a lower dimensional analog of this boson fermion model (as the TI surface host Dirac electrons) we need to use mass of the surface electrons in a similar calculation which is vanishingly small. Similar situation arise for a superconductor-graphene interface as well. In general, mass and charge of electronic quasi particle may differ by many orders from their vacuum value. Therefore in the cases when effective electron mass $m^* \sim \Lambda \sim \xi_\lambda$ and in presence of unpaired fermions the Coulomb term may dominate at large distance, otherwise the Meissner term would be the dominant term.

We have also found the general forms of vortex-vortex and fermion-vortex interactions for this system. It should be possible to reduce our calculations to 2+1 dimensions and find the effective vortex-fermion, fermion-fermion, and vortex-vortex interactions in planar type II superconductors with unpaired electrons. It is also possible to consider temperature dependence of the coupling constants and investigate critical phenomena in the vortex-electron system in the dual picture presented in this paper. We leave these for future work.

Chapter 3

Fermion- 2-form interaction and fermion pairing

3.1 Introduction

The low energy, long wavelength properties of a condensed matter system can be captured in an effective field theory, Ginzburg-Landau theory being the original example. Effective field theories which describe topological states of quantum matter are topological field theories, or more generally, quantum field theories which include topological interaction terms. The application of topological quantum field theories to condensed matter systems has, in recent years, greatly improved our understanding of both [6, 40, 55, 136–139]. Although the two-form gauge field is ubiquitous in these theories of topological matter, the difficulties of coupling it to electrons has stood in the way of a deeper understanding of the field and its applicability in condensed matter physics.

This is most easily seen from the point of view of gauge symmetries. Vector gauge transformations, which can be called the fundamental or defining symmetry of the gauge theory of two-form fields, generalize the gauge transformations of electromagnetism to $B \to B + \mathrm{d}\beta$, under which the field strength $H = \mathrm{d}B$ remains unchanged. But unlike the U(1) symmetry of ordinary gauge theory, this appears to have no representation as a local unitary transformation of fermions, and the interaction above does not remain invariant under this transformation.

There is also a problem of nonlocality. Duality between B and vector fields, for the example of the Abelian Higgs model, appears only through the field strength as $H = {}^*(A + \mathrm{d}\phi)$. So one might be encouraged to try an interaction with fermions in the form $H \wedge {}^*j$ where j could be either the usual fermionic current or the axial current. But this is also not correct. The reason is that the duality relation, between B on one hand and (A, ϕ) on the other, is not local. Thus in presence of fermions, the actions obtained by replacing one set of fields with the other are not equivalent and do not lead to the same equations of motion. The nonlocality can be understood if we remember that higher form gauge fields is that extended objects rather than point particles are essential to their definition. For example, the two-form B couples to worldsheets of strings rather than worldlines of particles, so one expects that the interaction between B and fermionic fields representing point particles could involve nonlocality in some form.

Such an interaction was proposed in [59] between a two-form and a nonlocal pseudotensor current J related to the curl of the fermion current

$$J = m\,{}^*\mathrm{d}\left(\Box^{-1} J_\psi\right) \equiv m^*\mathrm{d}\int d^4y\, G(x,y)\, J_\psi\,, \tag{3.1}$$

where J_ψ is the electron current, m is a mass scale appropriate to the problem where this interaction might be relevant, and $G(x,y)$ is the Green function of the wave equation,

$$\Box G(x,y) = \delta^4(x-y)\,. \tag{3.2}$$

This current is identically conserved, $\mathrm{d}^*J = 0$. The velocity field of the Dirac fermion is given by $v^i = \psi^\dagger\alpha^i\psi$, so the conserved charges $\Box J^{0i} = -m\epsilon^{ijk}\partial_j v_k$ can be identified with the vorticity field of the fermion. There is another way of looking at the conserved charges. In the absence of interactions, and in the non-relativistic limit in which the lower components of the Dirac fermions may be neglected for energies small compared to their mass, the static charge of the pseudovector current takes a simple form,

$$\left(J^{0i}\right)_{\mathrm{NR}} \propto \frac{1}{2}\left(\psi^\dagger\sigma^i\psi\right)\,. \tag{3.3}$$

The quantity on the right hand side is the spin density of the electron field, or the intrinsic magnetic moment density because it is multiplied by the electron charge. The proportionality becomes equality if m is chosen to be the electron mass. Just as the interaction between charges and currents is mediated by the 1-form gauge field A, the interaction between magnetic moments and their currents is mediated by the 2-form gauge field B.

We can write an action for the electrons and the gauge fields incorporating this interaction,

$$S = \int\left[\bar\psi\left(i\partial\!\!\!/ + eA\!\!\!/\right)\psi - m\bar\psi\psi + gB\wedge{}^*J - \frac{1}{2}F\wedge{}^*F + \frac{1}{2}H\wedge{}^*H\right]\,. \tag{3.4}$$

This action is invariant under the vector gauge transformations $B \to B + \mathrm{d}\beta$. We should think of this as a low energy effective action, valid for energy scales well below some cutoff Λ. The number of degrees of freedom can be worked out by first making the action local using Lagrange multipliers. The gauge field A has two degrees of freedom as usual, while B carries only one, since B_{0i} are non-dynamical and the vector gauge symmetry takes away two more degrees of freedom (not three, since β and $\beta + \mathrm{d}\chi$ are equivalent gauge parameters). There are two additional degrees of freedom in the Lagrange multipliers which stay around and may be thought of as auxiliary field degrees required for a local formulation of the non-local spin gauge interaction [59]. By formally treating the nonlocal interaction like any other coupling term, we can integrate out the fermions to find that the one-loop action contains a $B\wedge F$ interaction.

Such a term corresponds to topologically massive gauge theory, so the potential between two sources interacting via the gauge field ought to be Yukawa in the non-relativistic limit. But that is not what happens in this case. Calculated directly from the action in Eq. (3.4), the potential has two components. One is an r^{-1} Coulomb potential which corresponds to a massless gauge field, and the other is a linear potential which is attractive irrespective of the charges or spin alignment of the fermionic sources [60]. In this chapter we argue that the action of Eq. (3.4) can arise in a system described by local fields.

Since a linear potential is produced by a string, we start with a system which contains string-like objects as well as fermions. As we will see below, the nonlocal interaction appears naturally if we consider Abrikosov-Nielsen-Olesen (ANO) vortex strings in the Abelian Higgs model interacting with charged fermions. Such models containing coexisting bosonic and fermionic excitations are called boson-fermion models and are applicable to a range of phenomenon in condensed matter physics as we have described in Chapter 2. Below we give a very brief acount of our work.

In Sec. 3.2 we will show how the nonlocal interaction between fermions and the 2-form gauge field B arises in the dual picture of ANO strings interacting with fermions via the electromagnetic gauge field, however the action is not exactly what we are looking for. In Sec. 3.3 we calculate the static potential between two fermions and find that it also is not what we were hoping for, a mass term for the B field prevents the appearance of a linear potential. However, we show in Sec. 3.4 in a phase where the B field is massless, we recover Eq. (3.4) and thus the fermions are bound by a linear potential. We end with a discussion on possible physical implications of our results.

3.2 ANO string, dual variables, and currents

In the dual formulation of ANO strings in the Abelian Higgs model, the phase of the scalar field is written as the sum of singular and regular parts, then the singular field is dualized to the worldsheet of the string and the regular field is dualized to a 2-form. We have already shown the derivation of the dual theory in Chapter 2. Here we shall show a different route of reaching to the same action starting from the relativistic boson-fermion mixture model. As the dualization of the scalar field will be same here as well we shall skip that part, but the method here differs from that in Chapter 2 in the derivation of 2-form fermion non local coupling term. In Chapter 2 the 2-form- fermion non-local coupling comes as we integrate out photon field from the theory and replace the fermion-fermion interaction by QED. Here we shall dualize the electromagnetic part of the theory, in particular the kinetic term of photon field will be targeted. As we shall see this route of derivation will lead us to a dual photon or magnetic photon field which will be use full in our subsequent understanding of the problem at hand.

We start from the action [1]

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + |D_\mu \Phi|^2 + V(|\Phi|^2) + \bar{\psi}(i\partial\!\!\!/ - eA\!\!\!/ - m)\psi \right) , \tag{3.5}$$

where Φ is a complex scalar with electric charge q and ψ is a fermion with charge e. We have not assumed any relation between the charge e of the fermion and the charge q of the scalar. The potential $V(|\Phi|^2)$ has a degenerate minimum at $\Phi^*\Phi = v^2$ for some non-vanishing v^2, but the exact form of V is not important for our calculations. Vortex strings or magnetic flux tubes form in this model when the global U(1) symmetry is broken in the vacuum and the phase of Φ, which lives on the vacuum manifold $\Phi^*\Phi = v^2$, becomes multivalued. The position of the ANO string is the locus of the zeroes of Φ.

The corresponding partition function

$$Z = \int \mathscr{D}A_\mu \mathscr{D}\Phi \mathscr{D}\Phi^* \mathscr{D}\psi \mathscr{D}\bar{\psi} \; \exp iS \tag{3.6}$$

[1] While it is more economical to use the form notation, the calculations in this section and the next are more transparent in the index notation.

can be rewritten in presence of these vortex string configurations by using a polar decomposition of the complex scalar in field space as $\Phi = \frac{1}{\sqrt{2}}\rho \exp(i\theta)$. Then in the presence of flux tubes, we can write $\theta = \theta^r + \theta^s$, whereθ^s corresponds to a given magnetic flux tube, and θ^r describes single valued fluctuations around this configuration. For a string with winding number n , θ^s changes by $2\pi n$ for going around the string once, while ρ vanishes along the core of the string. The ANO string world sheet Σ is the collection of singular points of θ ,

$$\epsilon^{\mu\nu\lambda\rho}\partial_\lambda\partial_\rho\theta = \Sigma^{\mu\nu}(x) = 2\pi n \int d^2\sigma\, \epsilon^{ab}\frac{\partial X^\mu}{\partial\sigma^a}\frac{\partial X^\nu}{\partial\sigma^b}\delta^4(x - X(\sigma))\,, \tag{3.7}$$

where we have included the vorticity quantum 2π and the winding number n in the definition of the world sheet. The string carries quantized magnetic flux,

$$\oint_\Gamma A_\mu dx^\mu = \frac{2n\pi}{q}\,. \tag{3.8}$$

Now following the dualization procedure for the phase part of the scalar field, as was described in Chap. 2, we land up at the following dual lagrangian.

$$\begin{aligned}&\int \mathscr{D}A_\mu \mathscr{D}f\mathscr{D}\theta^s\mathscr{D}\bar{\psi}\mathscr{D}\psi\mathscr{D}B_{\mu\nu}\\ &\exp\left(i\int d^4x\Big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}v^2\partial_\mu f\partial^\mu f + \frac{1}{12f^2}H^{\nu\rho\lambda}H_{\nu\rho\lambda} - \frac{v}{2}\varepsilon^{\mu\nu\rho\lambda}B_{\rho\lambda}\partial_\mu\partial_\nu\chi^s\right.\\ &\left. - \frac{qv}{2}\varepsilon^{\mu\nu\rho\lambda}\partial_\nu B_{\rho\lambda}A_\mu - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + V(f^2) + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - eA_\mu\bar{\psi}\gamma^\mu\psi\Big)\right)\end{aligned} \tag{3.9}$$

One can now replace the functional integral over multivalued part of phase θ^s by the integral over all possible configuration of the world sheet $\tilde{x}^\mu(\sigma)$ of the effective string to make the dual theory an effective string theory and therefore fluctuation of the scalar field variable can be dealt with the string theoretic description of the model. In general, the world sheet of the effective string can be thought of as a surface in space-time where the scalar field becomes zero and thus can be expressed as the solution of the following equations

$$\mathrm{Re}\,\phi(\tilde{x}^\mu(\sigma)) = 0, \quad \mathrm{Im}\,\phi(\tilde{x}^\mu(\sigma)) = 0 \tag{3.10}$$

Now to consider such configuration into acount we need to integrate over the scalar field configuration which are regular everywhere except for the surface on which the above equations will be satified. In the polar coordinate the two dimensional manifold is determined by the singularities of the θ field and it should have the form given by Eq. (3.7). One can define the Jacobian of the transforming the field variable to the coordinate of the worldsheet $\tilde{x}(\sigma)$ parametrized by $\sigma_1 = \sigma, \sigma_2 = \tau$ with the help of Eq. (3.7) and is given by the following equation

$$J^{-1}[\Sigma_{\mu\nu}] = \int_\Sigma \mathscr{D}\tilde{x}\delta\left(\Sigma^{\mu\nu}(x) - 2\pi n\int_\Sigma d\sigma^{\mu\nu}(x(\xi))\,\delta^4(x - x(\xi))\,.\right), \tag{3.11}$$

where $d\sigma^{\mu\nu}(x(\xi)) = d^2\sigma\,\epsilon^{ab}\frac{\partial X^\mu}{\partial\sigma^a}\frac{\partial X^\nu}{\partial\sigma^b}$. This Jacobian was calculated in [42, 43, 140] and it was shown that a Nambu-Goto type action for the effective strings is obtained along with many other terms. In particular in [42] authors have taken the London limit for the AHM and have shown that for spherical topology of the world sheet the Jacobian would look like the following

$$J(\tilde{x}) = \exp\left[\mu \int_{\Sigma} d^2\sigma\sqrt{g} + \frac{11}{48\pi}\int d^2\sigma\left(\partial_a \ln\sqrt{g}\right)^2 + \beta\int d^2\sigma\sqrt{g}\left(\partial_a t_{\mu\nu}\right)^2\right]. \tag{3.12}$$

The terms other than the Nambu-Goto term were claimed to cancel the anomalous terms in the quantum version of Virasoro algebra in $D = 4$ dimension [42]. Therefore such strings can be quantized in $D = 4$ dimension. After transforming the variable from θ^s to $\tilde{x}(\sigma)$ we get the following generating functional

$$\begin{aligned}&\int \mathscr{D}A_\mu \mathscr{D}f\mathscr{D}\tilde{x}J(\tilde{x})\mathscr{D}\bar{\psi}\mathscr{D}\psi\mathscr{D}B_{\mu\nu}\\ &\exp\left(i\int d^4x\Big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}v^2\partial_\mu f\partial^\mu f + \frac{1}{12f^2}H^{\nu\rho\lambda}H_{\nu\rho\lambda}\right.\\ &\left.-\frac{v}{2}\varepsilon^{\mu\nu\rho\lambda}B_{\rho\lambda}\partial_\mu\partial_\nu\chi^s - \frac{qv}{2}\varepsilon^{\mu\nu\rho\lambda}\partial_\nu B_{\rho\lambda}A_\mu - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + V(f^2) + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - eA_\mu\bar{\psi}\gamma^\mu\psi\Big)\right)\end{aligned} \tag{3.13}$$

In the absence of strings we could eliminate $B_{\mu\nu}$ in favor of A_μ by summing over a perturbation series or equivalently by using the equations of motion, leaving only a massive gauge field [57, 141]. We cannot do that here because of the $B_{\mu\nu}\Sigma^{\mu\nu}$ interaction. So in order to diagonalize the system and eliminate the mixed $A - B$ term, we dualize the Maxwell electrodynamics as follows. The relevant part of the partition function is

$$\int \mathscr{D}A_\mu \exp i\int d^4x\Big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{6}q\varepsilon^{\mu\nu\rho\lambda}A_\mu H_{\nu\rho\lambda} - A_\mu\bar{\psi}\gamma^\mu\psi\Big). \tag{3.14}$$

Into this we insert the following Gaussian integral

$$N\int \mathscr{D}\chi_{\mu\nu}\exp\left(-\frac{i}{4}\int d^4x\left\{\chi_{\mu\nu}\chi^{\mu\nu} - \varepsilon^{\mu\nu\rho\lambda}\chi_{\mu\nu}F_{\rho\lambda} + \frac{1}{4}\left(\varepsilon^{\mu\nu\rho\lambda}F_{\rho\lambda}\right)^2\right\}\right) = 1, \tag{3.15}$$

where N is a constant (field-independent) normalization factor which is to lumped with other similar factors coming from other integrations. We now integrate over A_μ to be left with $\delta\left(\frac{1}{2}\varepsilon^{\mu\nu\rho\lambda}\partial_\nu\left(\chi_{\rho\lambda} - qB_{\rho\lambda}\right) - e\bar{\psi}\gamma^\mu\psi\right)$, which can be resolved by setting

$$\chi_{\mu\nu} = \partial_\mu A^m_\nu - \partial_\nu A^m_\mu + qB_{\mu\nu} + e\varepsilon_{\mu\nu\rho\lambda}\partial^\rho\frac{1}{\Box}\bar{\psi}\gamma^\lambda\psi, \tag{3.16}$$

where we have introduced another vector field A^m_μ, known as the magnetic photon. The partition function now has the form

$$\begin{aligned}\mathcal{Z} = &\int \mathscr{D}A^m_\mu\mathscr{D}\rho J(\tilde{x})\mathscr{D}\tilde{x}^\mu\mathscr{D}\psi\mathscr{D}\bar{\psi}\mathscr{D}B_{\mu\nu}\\ &\exp i\int d^4x\left[\frac{1}{2}\partial_\mu\rho\partial^\mu\rho + \frac{1}{12\rho^2}H^{\mu\nu\lambda}H_{\mu\nu\lambda} - \frac{1}{2}B_{\rho\lambda}\Sigma^{\rho\lambda} + V(\rho^2) + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi\right.\\ &\left.-\frac{1}{4}\Big(\partial_\mu A^m_\nu - \partial_\nu A^m_\mu + qB_{\mu\nu}\Big)^2 - \frac{1}{2}eqB^{\mu\nu}\varepsilon_{\mu\nu\rho\lambda}\partial^\rho\frac{1}{\Box}\bar{\psi}\gamma^\lambda\psi - \frac{1}{4}e^2\Big(\varepsilon_{\mu\nu\rho\lambda}\partial^\rho\frac{1}{\Box}\bar{\psi}\gamma^\lambda\psi\Big)^2\right].\end{aligned} \tag{3.17}$$

Here we notice that the last term in the integrand above can also come from integrating over the gauge field A_μ in the partition function of ordinary quantum electrodynamics with no additional field,

$$\int \mathscr{D}A_\mu \mathscr{D}\psi \mathscr{D}\bar{\psi} \exp i \int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 - eA_\mu \bar{\psi}\gamma^\mu \psi \right). \tag{3.18}$$

Thus we can reinstate the QED part of the action to write the action as

$$\boxed{\begin{aligned}&\int d^4x \Big[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - eA_\mu\bar{\psi}\gamma^\mu\psi - \frac{1}{2}eMB^{\mu\nu}\varepsilon_{\mu\nu\rho\lambda}\partial^\rho \frac{1}{\Box}\bar{\psi}\gamma^\lambda\psi - \frac{1}{4}\Big(\partial_\mu A^m_\nu - \partial_\nu A^m_\mu + MB_{\mu\nu}\Big)^2 \\ &+ \frac{1}{2}\partial_\mu\rho\partial^\mu\rho + \frac{v^2}{12\rho^2}H^{\nu\rho\lambda}H_{\nu\rho\lambda} - \frac{v}{2}B_{\rho\lambda}\Sigma^{\rho\lambda} + V(\rho^2) + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi \, , \Big]\end{aligned}} \tag{3.19}$$

where we have rescaled $B_{\mu\nu} \to vB_{\mu\nu}$, written $M = qv$, and suppressed the gauge-fixing term. The nonlocal interaction between $B_{\mu\nu}$ and charged fermions has appeared as a consequence of having ANO strings in the system.

Now we see that once we redefine $B_{\mu\nu} \to B_{\mu\nu} + \frac{1}{q}\left(\partial_\mu A^m_\nu - \partial_\nu A^m_\mu\right)$ the 4th term in the theory appear as a mass term for B field in the theory. In the alternative derivation of the dual action [142] in Chap. 2 it appears not in this form but as the equivalent nonlocal Meissner term $\frac{q^2v^2}{12}H^{\nu\rho\lambda}\frac{1}{\Box}H_{\nu\rho\lambda}$. There is an advantage of writing the mass term in the non-local form. The action of Eq. ((3.19)) which was found by dualization was invariant under the vector gauge transformation $B_{\mu\nu} \to B_{\mu\nu} + \partial_{[\mu}\Lambda_{\nu]}$. Written in the non-local form, the mass term and thus the action based on the Lagrangian of Eq. (3.19) remains invariant under the same transformation. In this paper we will work with the mass term in the local form as in Eq. (3.19), but we could have worked as well with the nonlocal mass term. Another way of handling the nonlocal mass term is by introducing an additional vector field [59], but we will not take that route in this chapter.

However, the redefinition of the 2-form gauge field changes the $B-$Fermion interaction term, $B - \Sigma$ interaction term as

$$\begin{aligned} S = \int d^4x \Big[&-\frac{1}{2}eM\left(B^{\mu\nu} - \partial^\mu A^{m\nu} + \partial_\nu A^{m\mu}\right)\varepsilon_{\mu\nu\rho\lambda}\partial^\rho\frac{1}{\Box}\bar{\psi}\gamma^\lambda\psi \\ &-\frac{v}{2}\left(B^{\mu\nu} - \partial^\mu A^{m\nu} + \partial^\nu A^{m\mu}\right)\Sigma_{\mu\nu}\Big]. \end{aligned} \tag{3.20}$$

Thus, after an integration by parts, we see that the magnetic photon couples to the the current $\partial^\nu\left(v\Sigma_{\mu\nu} + eM\varepsilon_{\mu\nu\rho\lambda}\partial^\rho\frac{1}{\Box}\bar{\psi}\gamma^\lambda\psi\right)$. As the magnetic photon does not have any kinetic part it only seems to be a Lagrange multiplier and the functional integral over A^m_μ would reveal

$$\int \mathscr{D}A^m_\mu \exp\left(i\int d^4x A^m_\mu \partial^\nu\left(v\Sigma_{\mu\nu} + \frac{1}{2}eM\varepsilon_{\mu\nu\rho\lambda}\partial^\rho\frac{1}{\Box}\bar{\psi}\gamma^\lambda\psi\right)\right) = \delta\left(\partial^\nu\left(v\Sigma_{\mu\nu} + eM\varepsilon_{\mu\nu\rho\lambda}\partial^\rho\frac{1}{\Box}\bar{\psi}\gamma^\lambda\psi\right)\right) \tag{3.21}$$

The delta function suggests that the combination $\left(v\Sigma_{\mu\nu} + \frac{1}{2}eM\varepsilon_{\mu\nu\rho\lambda}\partial^\rho\frac{1}{\Box}\bar{\psi}\gamma^\lambda\psi\right)$ is a constant independent of space-time. Therefore we get

$$\frac{1}{q}\Sigma_{\mu\nu} = -e\varepsilon_{\mu\nu\rho\lambda}\partial^\rho\frac{1}{\Box}\bar{\psi}\gamma^\lambda\psi \tag{3.22}$$

This equation suggests that the both the fermionic current and vorticity tensor are conserved in such a way that there is a correspondence between these two. As the left side of above equation represents magnetic field due to the fermionic source it seems that vortex lines would correspond to the magnetic field lines. More intuitive picture can be derived if we consider the special case $\mu, \nu = 0, z$ i.e. vorticity along z axis i.e. rotation of the superfluid on xy plane. In such a case we have

$$\frac{1}{q}\Sigma_{0z} = -e\varepsilon_{ik}\partial^i \frac{1}{\Box}\bar{\psi}\gamma^k\psi. \tag{3.23}$$

Now we write fermionic current using Gordon's decomposition

$$J^\mu = \frac{i}{2m}\left(\bar{\psi}\partial^k\psi - \partial^k\bar{\psi}\psi\right) + \frac{1}{m}\partial_\nu\left(\bar{\psi}\Sigma^{\nu k}\psi\right). \tag{3.24}$$

and thus get from Eq. (3.23)

$$\frac{1}{q}\Sigma_{0z} = -\frac{e}{m}\bar{\psi}\Sigma^3\psi - \frac{e}{m}\frac{1}{\nabla^2}\partial_i\partial_3\left(\bar{\psi}\Sigma^i\psi\right) + \frac{ie}{2m}\frac{1}{\nabla^2}\left[\vec{\nabla}\times\left(\bar{\psi}\vec{\nabla}\psi - \vec{\nabla}\bar{\psi}\psi\right)\right]_z. \tag{3.25}$$

As $\Sigma^3 = \mathrm{diag}(\sigma^3, \sigma^3)$, σ being the Pauli matrices, we see that in the non relativistic limit for fermions we have

$$\Sigma_{0z} \propto -\psi^\dagger\sigma^3\psi \tag{3.26}$$

This equation suggests that if we consider static configuration of a vortex along z axis the vortex become attached to the spin magnetic moment of the elelctron. Vorticity is also non-zero at points where magnetic field due to spin or charge current of electron is non-zero. This shows that the electrons, with a certain spin polarization, inside a superconductor would be attached to vortices due to their spin. This is quite similar to the flux attachment in the context of FQHE and anyons. We shall explore this connection in the next chapter. However, this attachment is also very important for fermion pairing as we shall see later in this chapter.

3.3 Effective fermion interaction

The nonlocal "spin-gauge" coupling between the fermions and the B-field was shown to give rise to a linear attractive potential between fermions, irrespective of whether they were positively or negatively charged [60]. However, as compared to the action considered there, several additional terms have appeared in the derivation from the boson-fermion model. The effective static interaction potential between nonrelativistic fermions in this system was calculated recently [142] by integrating out the gauge fields A_μ and $B_{\mu\nu}$. It is worthwhile to revisit this calculation, as we will be concerned in this paper with a particular modification of the result found there.

To find the effective static potential between non-relativistic electrons, we first integrate over the gauge fields A_μ and $B_{\mu\nu}$ using the Lagrangian of Eq. (3.19) with $\rho = v$. Introducing a gauge-fixing term we can integrate over A_μ,

$$\int \mathscr{D}A_\mu \exp i\int d^4x\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2 - eA_\mu\bar{\psi}\gamma^\mu\psi\right] \sim \exp\frac{i}{2}\int d^4k\ J^\mu(-k)\,\frac{1}{k^2}\,J_\mu(k)\,, \tag{3.27}$$

where J^μ is the fermion current and we have used the fact that it is conserved, $\partial_\mu J^\mu = 0$. For the B integration, we get

$$\int \mathscr{D}B_{\mu\nu} \exp\, i \int d^4x \left(-\frac{1}{4} B_{\mu\nu} K^{\mu\nu\rho\lambda} B_{\rho\lambda} - \frac{1}{2} B_{\mu\nu} J^{\mu\nu} \right) , \tag{3.28}$$

where we have written

$$K^{\mu\nu\rho\lambda} = \frac{1}{2} \left(\Box + M^2 \right) g^{\mu[\rho} g^{\lambda]\nu} + \frac{1}{2} \left(g^{\nu[\rho} g^{\lambda]\sigma} \partial_\sigma \partial^\mu - g^{\mu[\rho} g^{\lambda]\sigma} \partial_\sigma \partial^\nu \right) , \tag{3.29}$$

and

$$J^{\mu\nu} = v\Sigma_{\mu\nu} - eM\varepsilon_{\mu\nu\rho\lambda} \partial^\rho \frac{1}{\Box} \bar{\psi} \gamma^\lambda \psi \,. \tag{3.30}$$

The inverse of $K^{\mu\nu\rho\lambda}$ is the propagator for $B_{\mu\nu}$ and is given in momentum space as

$$G_{\mu\nu\rho\lambda}(k) = -\frac{1}{(k^2 - M^2)} \left(g_{\mu[\rho} g_{\lambda]\nu} - \frac{1}{M^2} \left(g_{\mu[\rho} k_{\lambda]} k_\nu - g_{\nu[\rho} k_{\lambda]} k_\mu \right) \right) , \tag{3.31}$$

where

$$K^{\mu\nu\rho\lambda} G_{\mu\nu\rho'\lambda'} = \left(\delta^\rho_{\rho'} \delta^\lambda_{\lambda'} - \delta^\rho_{\lambda'} \delta^\lambda_{\rho'} \right) . \tag{3.32}$$

Thus the integration over $B_{\mu\nu}$ will result in

$$\begin{aligned} &\int \mathscr{D}B_{\mu\nu} \exp \left(i \int d^4x \Big(-\frac{1}{4} B_{\mu\nu} K^{\mu\nu\rho\lambda} B_{\rho\lambda} - \frac{1}{2} B_{\mu\nu} J^{\mu\nu} \Big) \right) \\ &\qquad \sim \exp \left(\frac{i}{2} \int d^4k\, e^2 M^2\, J^\mu(-k) \left[\frac{1}{k^2(k^2 - M^2)} \right] J^\mu(k) + \ldots \right) \end{aligned} \tag{3.33}$$

where we have used the expression for $J^{\mu\nu}$ in terms of the fermion current J^μ. The dots stand for terms involving the string worldsheet Σ [142]. To calculate the interaction potential in the static limit one need to derive the scattering amplitude for a electron-electron Møller scattering and in this case it is given by the following expression

$$\mathcal{M} \sim (\bar{u}_{q_2} \gamma^\mu u_{q_1}) \frac{e^2 M^2}{k^2(k^2 - M^2)} (\bar{u}_{p_2} \gamma_\mu u_{p_1}) , \tag{3.34}$$

where q_2, q_1 and p_2, p_1 are the initial and final momentum of the electrons which are taking part in the scattering and $k = p_2 - p_1 = q_2 - q_1$ is the momentum exchanged between two electrons. This suggests that the static potential will be

$$V(r) \sim \int d^3k e^{-i\vec{k}\cdot\vec{r}} \frac{1}{|\vec{k}|^2(|\vec{k}|^2 + M^2)}, \tag{3.35}$$

where in the non relativistic limit we have neglected the frequency part i.e. $k^2 \simeq |\vec{k}|^2$, $\omega \sim 0$. As one can write $\frac{1}{k^2(k^2 - M^2)} = \left(\frac{1}{(k^2 - M^2)} - \frac{1}{k^2} \right)$, it seems that the net potential will be only the screened Coulomb or "Yukawa potential". But at this point we need to be careful as the gauge field $B_{\mu\nu}$ is an emergent gauge field and the integration over 4-momentum of the propagator of this gauge field must not be extended from 0 to ∞. Rather, there must exist an upper limit to the momentum amplitude which corresponds to coherence length of the superconductor and also have a lower limit fixed by the system size. Therefore the net force would not be a screened Coulomb in general. However, in the London limit $\sqrt{\lambda} \to \infty$ and for a large system the limit can be set approximately to: 0 to ∞. Thus we see that, in this limiting case, the $\frac{1}{k^2}$ appearing due to the Coulomb potential cancels with a negative term appearing after the integration of the B field so that the propagator corresponds to the Yukawa potential $\frac{\exp(-Mr)}{r}$. As the interaction is screened it is therefore not possible to propagate a longrange force like $V(r) \sim r$. Thus we see no linear component would appear in this limit.

At this point we need to think about the electron-vortex attachment and the length scalaes of AHM. As we know there are two different length scales, one is the penetration depth $\xi_e = 1/(ev)$ and another is the the coherence length $\xi_\lambda = 1/(\sqrt{\lambda}v)$, λ being the coupling constant of ϕ^4 term in AHM. Above the scale ξ_e the magnetic field becomes completely screened and thus for a vortex one can appreciate that the magnetic field lines would not exist beyond this distance. On the other hand below ξ_λ the value of vacuum expectation value of scalar field starts to decrease sharply and at the core of the vortex it becomes zero. Therefore beyond this length scale superconductivity cease to exist and it would be difficult to discuss on this dual theory. For type II superconductors $\xi_e > \xi_\lambda$ and for this reason there exist a region $\xi_\lambda < r \leq \xi_e$ where $V \neq 0$ and also magnetic field lines still exist. Thus for two electronic modes trapped into this region, magnetic interaction may still exist.

Let us now imagine a collection of electrons inside (or in proximity to) a superconducting condensate when vortices are allowed to form. As spin magnetic moment of electrons attach to vorticity each electron will be trapped in a vortex line. Therefore the magnetic field lines due to spin would vanish outside the magnetic petetration depth. Now, if we bring two of these vortex attached electrons so close to one another that the characteristic length ξ_e of each flux tube overlap, then the electron magnetic moment start to interact. In such a situation, the flux tube attached to one electron, whose end points otherwise extends to the boundary of the system, ends on another electron and thus two such electrons become bound to to each other by two flux tubes.

This is quite similar in spirit to the dual superconductor scenario of quark confinement [143], but different in several aspects. For monopoles, the flux lines are isotropic in ordinary media. Thus when two such monopoles with opposite charges are put into a superconductor the only configuration in which magnetic field does not pass through the condensate is that in which two monopoles become connected by a flux tube. Once such a pair is constructed, can not be broken when monopoles are taken far apart. But for dipoles, it is possible that the two poles connects to two flux tube and the flux tubes extend to the boundary of the system. Thus the configuration two dipoles connected by two flux tubes seems to exist only when the electrons are at a separation $r \leq \xi_e$ and reduces to the configuration of individual flux attached elelctrons when they are taken far apart as the later configuration becomes energetically more favourable.

When the configuration of two dipoles connected by flux tubes forms, the electrons should feel a linear potential acting between them as the energy of the configuration inceases linearly with the length of the flux tubes. A verification of this statement can also be shown from Eq. (3.33) by expanding $1/k^2(k^2 - M^2)$ as

$$\frac{1}{k^2(k^2 - M^2)} \simeq \frac{1}{k^4} - \frac{M^2}{k^6} + ..., \tag{3.36}$$

where we have taken the approximation that $k^2 < M^2$ which is possible when inter

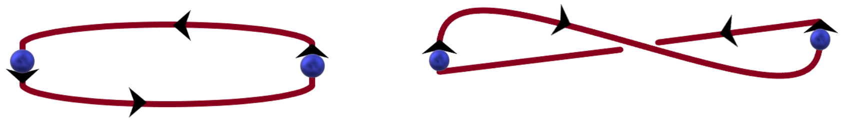

Figure 3.1: A pair of electrons connected by flux tubes. The arrows indicate the direction of magnetic flux.

particle separation $r \leq \xi_e$. As we can recognize, this is the case we discussed above. Therefore for $\xi_\lambda < r \leq \xi_e$ the Eq. (3.33) gives

$$\int \mathscr{D}B_{\mu\nu} \exp\left(i\int d^4x\left(-\frac{1}{4}B_{\mu\nu}K^{\mu\nu\rho\lambda}B_{\rho\lambda} - \frac{1}{2}B_{\mu\nu}J^{\mu\nu}\right)\right)$$
$$\sim \exp\left(\frac{i}{2}\int d^4k\, e^2M^2\, J^\mu(-k)\left[\frac{1}{k^4}\right]J^\mu(k) + ...\right). \tag{3.37}$$

Thus following the prescription outlined in Sec. 3.3 the momentum space version of interparticle potential (under our approximation) can be written as

$$V(|\vec{k}|) = -e^2\frac{1}{|\vec{k}|^2} + e^2M^2\frac{1}{|\vec{k}|^4}. \tag{3.38}$$

To achieve the configuration space version of this potential we need to do a Fourier transform. But for the second term we need to be careful as it is only approximately valid for $M < |\vec{k}| < \sqrt{\lambda}v$. Implementing this approximation we have

$$V(r) = -e^2\int d^3k\ e^{\vec{k}\cdot\vec{r}}\frac{1}{|\vec{k}|^2} + e^2M^2\int_{ev}^{\sqrt{\lambda}v} d^3k\ e^{\vec{k}\cdot\vec{r}}\frac{1}{|\vec{k}|^4}. \tag{3.39}$$

However, for $\sqrt{\lambda}v >> M$ the limits in the second term can be set approximately to be 0 to ∞ for all practical purposes. In this limit one gets from above equation

$$V(r) = e^2\frac{1}{r} + e^2M^2\, r. \tag{3.40}$$

If we now minimize this potential with respect to r then the we get the interparticle separation for a stable pair is $r = r_0 = \dfrac{1}{M}$, which gives a verification of the arguments we presented above.

For a critical superconductor, for which $\xi_e = \xi_\lambda$, the above arguments would fail. Even then one can realize the linear potential between two electrons in a phase where the mass term is not present. We shall consider this question, whether there can be such a phase, in the next section.

3.4 String Higgs mechanism and the mass of B

The photon becomes massive in the Anderson-Higgs mechanism which provides the phase transition from a massless gauge field to a massive one; the phase transition in the opposite direction is accompanied by symmetry restoration as the system is raised above the transition temperature. The 2-form $B_{\mu\nu}$ is a higher analogue of the usual 1-form gauge field A_μ, so what we have in mind is "a higher analogue" of spontaneous symmetry breaking. Since $B_{\mu\nu}$ couples to worldsheets of vortex strings, it is the condensation of these strings which should produce the Higgs mechanism for the B-field. The idea of a phase transition due to vortex condensation at finite temperature is not a new one, more generally examples are known in quantum field theory and condensed matter physics of phase transitions driven by condensation of topological defects [144–146].

3.4.1 Higgs mechanism for $B_{\mu\nu}$

In this section we try to summarize the idea of a Higgs mechanism for $B_{\mu\nu}$ following [123, 147–149]. Let us start by writing down the action for classical strings. For a point particle the proper length of the world line travelled by the particle is the ovserver independent action. Similarly for strings the quantity which remains independent of different observers is the proper area traversed by the string. Thus one may choose the action of a classical string to be proportional to this proper area and is called Nambu-Goto action. Let $\eta_{\mu\nu}$ be the metric on a $d+1$ dimensional space-time. Then the induced metric on the world sheet is given by

$$g_{ab} = \eta_{\mu\nu}\frac{\partial X^\mu}{\partial y^a}\frac{\partial X^\nu}{\partial y^b} \tag{3.41}$$

where $y^a = \sigma, \tau$ for $a = 1 or a = 2$. The invariant area element on the world sheet is given by

$$dA = d\sigma d\tau\sqrt{g}, \tag{3.42}$$

where g is the determinant of the induced metric and is given by

$$g^2 = \left(\dot{X}^2 X'^2 - (\dot{X}\cdot X')\right). \tag{3.43}$$

Thus the Nambu-Goto action for the classical string is

$$S_{N-G} \sim \int d\sigma d\tau\sqrt{g}. \tag{3.44}$$

If we now define $\Sigma_{\mu\nu} = X'_\mu\dot{X}_\nu - X'_\nu\dot{X}_\mu$ as the plaquette variable then the N-G action can be reexpressed in terms of these variables as

$$S_{N-G} \sim \int d\sigma d\tau\sqrt{\Sigma^{\mu\nu}\Sigma_{\mu\nu}}. \tag{3.45}$$

The 2-form gauge field couples to the plaquette element as electromagnetic vector potential couples to the current. Therefore the N-G string action modifies in presence of an external 2-form field $B_{\mu\nu}$ as

$$S = \int d\sigma d\tau\left[T\sqrt{\Sigma^{\mu\nu}\Sigma_{\mu\nu}} + gB_{\mu\nu}\Sigma^{\mu\nu}\right], \tag{3.46}$$

Here T is the static string tension. From the above equation we can derive the momentum variable conjugate to plaquette variable $\Sigma_{\mu\nu}$ as

$$P_{\mu\nu} = \frac{\delta S_{closed}}{\delta \Sigma_{\mu\nu}} = T\frac{\Sigma_{\mu\nu}(X)}{\sqrt{\Sigma^2}} - gB_{\mu\nu}(X). \tag{3.47}$$

This equation in turn gives us the on shell equation for the plaquette variables

$$(P_{\mu\nu} + gB_{\mu\nu})^2 = T^2. \tag{3.48}$$

This is similar to relativistic energy-momentum relation for point particles where particle momentum is replaced by this conjugate variable $P_{\mu\nu}$ and mass is replaced by string tension. In analogy to the relativistic point particles one can think of quantizing this on-shell equation for strings by defining a wavefunctional for strings $\Psi[\Omega]$ on the space of parametrized loops $\{\Omega = \{X^\mu(\sigma,\tau) = X^\mu(\sigma + 2\pi, \tau)\}\}$ which remains invariant under a reparametrization $\Omega(\sigma) \to \tilde{\Omega}(\tilde{\sigma})$,

$$\Psi[\tilde{\Omega}(\tilde{\sigma})] = \Psi[\Omega(\sigma)]\,. \tag{3.49}$$

and then replacing the conjugate momentum by its differential operator version $-i\dfrac{\delta}{\delta\Sigma^{\mu\nu}}$. Thus the quantum mechanical wave equation for Ψ can be written as

$$\left[-i\frac{\delta}{\delta\Sigma^{\mu\nu}} + gB_{\mu\nu}\right]^2 \Psi[\Omega] = T^2\Psi[\Omega]\,, \tag{3.50}$$

where we have written g for the coupling constant between $B_{\mu\nu}$ and the worldsheet. The physical significance of the wave functional $\Psi[\Omega]$ can be realized in the so called static parametrization : In a chosen Lorentz frame of reference take $X_0 = \tau$, $\vec{X} = \vec{X}(\tau,\sigma)$ with τ being the imaginary time. Then $\Psi[\Omega]$ represent the quantum amplitude of finding the string in the configuration $[X(\sigma,\tau),\ \sigma = (0, 2\pi)]$ at imaginary time τ.

The functional derivative with respect to $\Sigma_{\mu\nu}$ is called a plaquette derivative and defined as follows. For a given loop configuration Ω we may add a small plaquette element $\delta\Sigma = d\sigma\, d\tau \Sigma_{\mu\nu}$ at a point $X(\sigma) \in \Omega$, which induces a deformation in the normal direction of the string loop: $\Omega(\sigma) \to \Omega(\sigma) + \delta\Omega$ such that $\delta\Sigma = \delta\Omega$. Such a deformation of parametrized loop configuration induces a change in $\Psi[\Omega]$ as well. The change is expressed by the following equation

$$\Psi[\Omega + \delta\Omega] - \Psi[\Omega] = \int_0^{2\pi} \sqrt{h} d\sigma \frac{\delta\Psi[\Omega]}{\delta\Sigma_{\mu\nu}} \delta\Sigma_{\mu\nu}, \tag{3.51}$$

where $\sqrt{h}$ is the Jacobian of the transformation from $X^\mu(\sigma)$ to σ and is written as $h = \left[\dfrac{dX^\mu}{d\sigma}\right]^2$. This rate of change of string wave functional $\Psi[\Omega]$ with the deformation of the loop configuration is defined as Plaquette derivative.

For the second quantized string we consider a complex scalar functional $\Psi[\Omega]$ on the loop space, invariant under reparametrizations as before. Taking a clue from the quantum mechanics of the string, and using above described construction of plaquette derivative we write an action of this string field interacting with the B-field,

$$S = \int d^4x \frac{1}{12} H^2_{\mu\nu\lambda} - \int [dx(\cdot)] \oint \sqrt{h} d\sigma \left[\left|\frac{\delta\Psi[\Omega]}{\delta\Sigma^{\mu\nu}} + igB_{\mu\nu}\Psi[\Omega]\right|^2 + \mu^2|\Psi[\Omega]|^2\right] + S_{\text{int}}\,. \tag{3.52}$$

Here the integral over $[dx(\cdot)]$ represents summation over all possible configurations of parametrized string loops. The string interaction term S_{int} represents splitting and joining of strings through cubic, quartic and similar terms. There should be interaction terms between the string and the Higgs field as well. This action is invariant under a global U(1) transformation $\Psi[\Omega] \to e^{i\omega}\Psi[\Omega]$ and also under string reparametrizations $\Psi[\Omega(\sigma)] \to \tilde{\Psi}[\tilde{\Omega}(\tilde{\sigma})]$. Following [150], we gauge the U(1) transformation by making it local on loop space,

$$\Psi[\Omega] \to \Psi'[\Omega] = e^{ig\omega[\Omega]}\Psi[\Omega]\,, \qquad \omega[\Omega(\cdot)] = \oint_\Omega dx^\mu \Lambda_\mu\,. \tag{3.53}$$

Then the gauge covariant area derivative becomes

$$\frac{\delta\Psi'}{\delta\Sigma^{\mu\nu}} + igB'_{\mu\nu}\Psi' = e^{i\omega[\Omega]}\left[\frac{\delta}{\delta\Sigma^{\mu\nu}} + ig\left(B'_{\mu\nu} - \frac{1}{g}\partial_{[\mu}\Lambda_{\nu]}\right)\right]\Psi\,. \tag{3.54}$$

Thus the gauge covariant derivative transforms homogeneously and the action remains invariant provided $B_{\mu\nu}$ undergoes a vector gauge transformation as $B_{\mu\nu} \to B'_{\mu\nu} = B_{\mu\nu} + \partial_{[\mu}\Lambda_{\nu]}$.

The idea of a Higgs mechanism corresponding to this symmetry is a generalization of the usual one for Abelian gauge fields. The vortex loops condense into the vacuum for $\mu^2 < 0$ and the functional field gets a nonvanishing vacuum expectation value (vev). The simplest case is when this vev is a constant in spacetime, i.e. $\langle 0|\Psi[\Omega]|0\rangle = \frac{1}{2g}M_B$, in which case the Lagrangian for the "free" B-field becomes

$$\mathscr{L} = \frac{1}{12}H^2_{\mu\nu\lambda} - \frac{1}{4}M_B^2 B^2_{\mu\nu}\,. \tag{3.55}$$

This is thus exactly like the Higgs mechanism, but the gauge field which becomes massive is $B_{\mu\nu}$. We will call this the Higgs mechanism for strings, and distinguish it from the "usual Higgs mechanism", by which we will mean the Higgs field Φ getting a nonvanishing vev and the photon A_μ becoming massive. The main condition under which such a string field condensation happens is that μ^2 $, <$ 0. Such a condition is satisfied for a collection of line like extended objects was shown by D. Forster [151], S.J. Rey [149] and others. Below, in the next section, we shall review the essence of these works.

3.4.2 False vacuum for $\Psi[\Omega]$

In order to understand how entropy effects may lead to condensation of two dimensionally extended structures(in this case vortex strings), we calculate the Euclidean partition function [123, 147–149, 151]. Let us consider a hypercubic lattice in four Euclidean dimensions, with lattice spacing a. The $B_{\mu\nu}$ field couples to area elements, so it is defined in terms of the plaquette operator on a plaquette p as

$$U_p(B_{\mu\nu}) = \exp\Big[-iga^2 B_{\mu\nu}(p)\Big]\,. \tag{3.56}$$

The vector gauge transformation $B_{\mu\nu} \to B_{\mu\nu} + \partial_{[\mu}\Lambda_{\nu]}$ acts along the links on the boundary of a given plaquette p. Writing $\Lambda_l = \exp[-iga\Lambda_\mu]$ we find that the plaquette operator transforms under the gauge transformation as

$$U_p(B_{\mu\nu}) \to \left[\prod_{l\in\partial p}\Lambda_l\right] U_p(B_{\mu\nu})\,. \tag{3.57}$$

Then the gauge invariant kinetic term in Eq. (3.52) is the sum over all lattice cubes of the product of the plaquette operators residing on the boundary of each cube,

$$\int d^4x \frac{1}{12} H^2 = \beta \sum_{\text{cube}} \mathrm{Re} \left[\prod_{p \in \partial(\text{cube})} U_p \right] , \tag{3.58}$$

where the lattice coupling constant of the $B_{\mu\nu}$ field has been denoted by β .

For the "kinetic term" of the functional field $\Psi[\Omega]$ the effective Lagrangian is calculated from a sum over configurations

$$K(C_1, C_2, A) = e^{-\tau_s a^2 A} \sum_S \left[\prod_{p \in S} U_p(B_{\mu\nu}) \right] , \tag{3.59}$$

where the sum is over all Euclidean world sheets S of area A connecting the closed curves C_1 and C_2 , the bare string tension is τ_s , and the surface S is taken to be orientable and without holes.

Modifying C_2 by a keyboard-like plaquette variation at a link produces a recursion relation

$$K(C_1, C_2, A) = \sum_p \left[\bar{U}_p K(C_1, C_2 + p, A - a^2) + U_p K(C_1, C_2 - p, A - a^2) \right] , \tag{3.60}$$

where this sum is over all plaquettes which can be added at the given link. Since there are $2(d-1)$ such plaquettes in d dimensions, it follows that $K(C_1, C_2, A)$ satisfies the diffusion equation

$$\frac{\partial}{\partial \bar{A}} K(C_1, C_2, \bar{A}) = \left[\sum_p \left(\frac{\delta}{\delta \Sigma^{\mu\nu}} + igB_{\mu\nu} \right)^2 - M_0^2 \right] K(C_1, C_2, \bar{A}) \,, \tag{3.61}$$

where we have written $\bar{A} = a^2 A e^{-\tau_s a^2}$ and the dynamical string tension M_0 is related to the bare string tension τ_s as

$$M_0^2 = \frac{1}{a^4} (e^{\tau_s a^2} - 6) \tag{3.62}$$

in four spacetime dimensions. The propagator $G(C_1, C_2)$ of a closed string can be written as

$$G(C_1, C_2) = \int_0^\infty d\bar{A}\, K(C_1, C_2, \bar{A}) \,, \tag{3.63}$$

which can be obtained from the action

$$S(\Psi[\Omega], B_{\mu\nu}) = \int [dx(\cdot)] \oint \sqrt{h} d\sigma \left[\left| \frac{\delta \Psi[\Omega]}{\delta \Sigma^{\mu\nu}} + igB_{\mu\nu} \Psi[\Omega] \right|^2 - M_0^2 |\Psi[\Omega]|^2 \right] . \tag{3.64}$$

This is the same as the second term of Eq. (3.52) written in Euclidean space and with $\mu^2 = M_0^2$. Thus we see that it is possible that in the action Eq. (3.52) of the vortices, we may have $\mu^2 < 0$ for some system of the kind we are considering. The effective potential for Ψ will in general have contributions also from its interaction with $\Phi^\dagger \Phi$ and with the fermions, which will modify the expression for M_0^2 . The first kind of interaction appears to involve $(\Phi^\dagger \Phi)^{-1}$ when the strings are close to being stable, while the fermion interaction is nonlocal in the leading order.

3.4.3 Linear potential

We have seen in previous discussions that there exists such conditions under which extendended structures such as strings can also condense. Such condensation may give rise to non-zero vev of the string field $\Psi[\Omega]$ as a large number of strings would be present at the ground state. As the 2-form field couples minimally to world sheet of vortex lines, condensation of vortex lines gives rise to mass of the 2-form field in a analogous way to "Higgs mechanism" by breaking the local U(1) vector gauge symmetry of the Lagrangian. We have also seen that the dual action of Eq. 3.19 contains effective strings described by world sheet $\Sigma_{\mu\nu}$ coupling minimally to a 2-form gauge field. This 2-form gauge field is massive and the mass is defined by the parameters of the original theory i.e. $M = qv$. With the previous discussions it becomes obvious that this dual theory of effective on-shell strings can be generalized to a string field theory where strings would be described by quantum field $\Psi[\Omega]$. In such a theory the mass of the B field would come from a string field condensation or "Higgs mechanism of stings". Thus the generalized string field theoretic version of the dual action would be

$$S_{Dual} = \int d^4x \frac{1}{12} H^2_{\mu\nu\lambda} - \int [dx(\cdot)] \oint \sqrt{h} d\sigma \left[\left| \frac{\delta \Psi[\Omega]}{\delta \Sigma^{\mu\nu}} + igB_{\mu\nu}\Psi[\Omega] \right|^2 + \mu^2 |\Psi[\Omega]|^2 \right] + S_{\text{int}}$$
$$+ \int d^4x \left[-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - eA_\mu \bar{\psi}\gamma^\mu\psi - \frac{1}{2} eMB^{\mu\nu}\varepsilon_{\mu\nu\rho\lambda}\partial^\rho \frac{1}{\Box} \bar{\psi}\gamma^\lambda\psi + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi \right] \tag{3.65}$$

As we can appreciate, this generalized action have two ground states. One is characterized by $\mu^2 > 0$ and for this case the 2-form field is mass less. Whereas there is another phase characterized by $\langle 0|\Psi[\Omega]|0\rangle \neq 0$. In this vacuum strings condense and give rise to non-zero mass of 2-form field. It is that state where we recover our dual lagrangian where B field has mass $M_B = 4g^2\langle 0|\Psi[\Omega]|0\rangle^2$. Now we can think of a phase transition in the reverse direction i.e. from the true vacuum of the string field where B field is massive to the false vacuum where $B_{\mu\nu}$ field is mass less. In this false vauum state we get the following Lagrangian

$$\boxed{\begin{aligned} \mathscr{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{12}H^{\nu\rho\lambda}H_{\nu\rho\lambda} + \bar{\psi}(i\partial\!\!\!/ - m)\psi \\ &- e\bar{\psi}A\!\!\!/\psi - \frac{1}{2}\tilde{g}B^{\mu\nu}\varepsilon_{\mu\nu\rho\lambda}\partial^\rho\frac{1}{\Box}\bar{\psi}\gamma^\lambda\psi - \frac{1}{2}g_s B_{\rho\lambda}\Sigma^{\rho\lambda}\,, \end{aligned}} \tag{3.66}$$

where the fields and their couplings now include quantum corrections. We notice that the coefficient of B-fermion interaction term and that of mass term of B field in Eq. (3.19) were related and hence setting the mass of the B field to zero would also lead to vanishing contributions of the B-fermion interaction term. However in generalizing the dual theory into a string field theory (SFT) the coefficients of the above mentioned terms become dynamical and thus the relation of the coefficients may have spoiled. Thus setting $M_B = 0$ would not set the coefficient of B-fermion interaction term to zero in SFT version. We have assumed that the mass of ρ remains much bigger than the energies of thermal fluctuations, so in particular ρ is frozen. This is necessary for the system to remain in the state where strings can form.

We calculate the effective static potential between a pair of charged fermions by integrating out the gauge fields from the action as before, but now we will need a gauge-fixing term $\frac{1}{2\eta}(\partial_\nu B^{\mu\nu})^2$ because the B-field is massless. Then the B-propagator is

$$G^{\mu\nu\rho\lambda}(k) = -\frac{1}{k^2}\left(g_{\mu[\rho}g_{\lambda]\nu} - \frac{1-\eta}{k^2}\left(g_{\mu[\rho}k_{\lambda]}k_\nu - g_{\nu[\rho}k_{\lambda]}k_\mu\right)\right) . \tag{3.67}$$

Using this in the B-integration of Eq. (3.33), but with the coupling constants as in Eq. (3.66), we get

$$\int \mathscr{D}B_{\mu\nu} \exp\Bigg(i\int d^4x\Bigg(-\frac{1}{4}B_{\mu\nu}K_0^{\mu\nu\rho\lambda}B_{\rho\lambda} - \frac{1}{2}B_{\mu\nu}J^{\mu\nu}\Bigg)\Bigg)$$
$$\sim \exp\left(\frac{i\tilde{g}^2}{2}\int d^4k\ J^\mu(-k)\frac{1}{k^4}J^\mu(k) + ...\right), \tag{3.68}$$

where we have written $K_0^{\mu\nu\rho\lambda}$ for the kinetic operator with $M = 0$,

$$K_0^{\mu\nu\rho\lambda} = \frac{1}{2}g^{\mu[\rho}g^{\lambda]\nu}\Box + \frac{1}{2}\left(1 - \frac{1}{\eta}\right)\left(g^{\nu[\rho}g^{\lambda]\sigma}\partial_\sigma\partial^\mu - g^{\mu[\rho}g^{\lambda]\sigma}\partial_\sigma\partial^\nu\right) . \tag{3.69}$$

Thus from Eq. (3.68) we can conclude that effective static potential between electrons is then attractive and linear,

$$V(r) = \tilde{g}^2 r\,. \tag{3.70}$$

3.5 Discussion

It is well known that the Abelian Higgs model with flux strings has a dual description in terms of a two-form potential (sometimes called the disorder field in the context of superconducting phase transitions [152, 153]).In this chapter we have tried to explore the consequences of the effective nonlocal interaction of vortex lines with electrons, which emerges as a term in the dual action in Chap. 2.

In particular we argue that for a type II superconductor when the separation between two electrons r is : $\xi_\lambda < r \leq \xi_e$ then they become connected by flux tubes. Two electrons thus connected by effective strings may feel a linear attractive potential. The nonrelativistic interaction potential for two such electron trapped into the specific region and the size of a stable pair seems to support this claim. We have also discussed that there can be another phase where the spin polarized electrons connects to flux tubes which ends at the boundary of the system. These two phases would compete among each other and may lead to a phase transition from one to another when density of such itinerant spin polarized electrons are varied. The phase where electrons become connected by flux tubes may be called a“confined phase”, while the other phase in which individual electrons are connected by flux tubes may be called a “deconfined phase”.

Alternatively one can also relize the linear potential using $B_{\mu\nu}$-fermion non-local interaction term by mapping the dual action to a corresponding string field description where mass of the 2-form field, present in the dual action, can be interpreted as coming from the “string Higgs mechanism”. In the false vacuum state of the string field the mass of the 2-form field vanishes and a linear potential between the electrons would be realized. A linear potential between pairs of particles would be interpreted as a flux tube. The flux tubes we started out with are either infinite (ending on the system boundary) or end on magnetic charges [154], not electric charges. This problem can be resolved by remembering the picture of electronic magnetic dipoles connected by magnetic flux tubes (see 3.1) discussed previously and the linear potential seems to be an indication of the same phenomenon in this case as well.

A cartoon representing a semiclassical view of the electron pair is shown in Fig. 3.1, the north pole of each dipole connected to the south pole of the other by a flux tube. In our construction the dipoles are electrons, thus point dipoles, and the flux tubes are infinitesimally thin. The dipoles can be antiparallel or parallel as in the figure, or they can be oriented in any which way with respect to each other.

We note that this construction of fermions linked by vortex strings is different from another recent construction in which bosonic Cooper pairs are connected by electric flux tubes in a model of superinsulators [155–157]. Our construction here is different from that one in two important aspects. One is that electric flux strings appear in dual superconductivity, which exhibits a dual Meissner effect and excludes electric fields from the bulk, constricting the flux into the strings (in analogy to QCD confinement [143,158–160]). In our construction, we have the usual (type II) superconductivity arising from condensation of electric charges, which constricts the magnetic field into strings. The other difference is that in that model, electrically charged Cooper pairs are connected by the strings, the electric flux in a string ends on electric charges at the ends. In our construction, electrically charged fermions are connected by strings carrying magnetic flux, because charged fermions behave as point magnetic dipoles due to their spin. The magnetic field of a point dipole is constricted into a pair of strings in a superconductor, connecting it to another fermion as in Fig. 3.1. Bound states of three or more electrons are also possible, as shown in Fig. 3.3.

We can estimate the energy of a localized pair as a function of its angular momentum, if we pretend that the electrostatic potential between two electrons joined by a magnetic flux tube remains the usual Coulomb potential. For a nonrelativistic string of length L and tension T, with electrons of mass m and charge e at the two ends, the energy is then

$$E = 2m + TL + \frac{1}{2}I\omega^2 + \frac{e^2}{L}\,, \tag{3.71}$$

where I is the moment of inertia,

$$I = \frac{1}{12}TL^3 + \frac{1}{2}mL^2\,. \tag{3.72}$$

Writing $J = I\omega$ for the angular momentum, we find the relation between the energy E and the angular momentum J of the string

$$E = 2m + TL + \frac{6J^2}{L^2(TL+6m)} + \frac{e^2}{L} = 2m + E_b\,, \tag{3.73}$$

where we have defined E_b to be the binding energy. The same result can also be found by taking the non-relativistic limit of the relativistic string with massive end points [161]. Minimizing the energy with respect to the length, one can calculate L as a function of the other parameters. Putting this back into the equation for E we get a relation between the energy and the angular momentum. For type II superconductors the penetration depth is of the order ∼100 nm, which gives a representative string tension $T \sim 10$ eV^2. The corresponding plot of E_b vs J is shown in Fig. 3.2 . The size of the pair corresponding to the $J = 0$ state is ∼19 nm. Bound states of three or more electrons are also possible, as

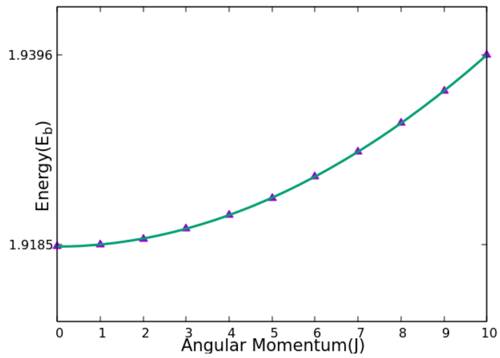

Figure 3.2: E_b (in eV) vs J (in $\hbar$) plot

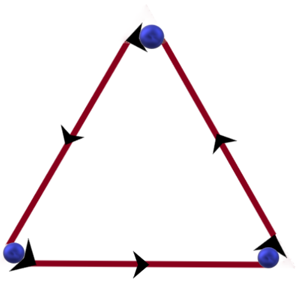

Figure 3.3: Three electrons connected by magnetic flux tubes.

shown in the example of Fig. 3.3. The energies are higher – for example, $E_b = \dfrac{3}{2}TL + \dfrac{3e^2}{L}$ for the $J = 0$ state of three electrons.

The possibility of having different number of fermions in bound states, with different geometries, suggests that our construction can also be useful as a toy model of quark confinement. A flux string in the Abelian Higgs model can end on a magnetic monopole, leading to magnetic confinement. The usual string picture of quark confinement [143,160] is the dual of this, where the vacuum is thought of as a dual (color) superconducting vacuum in which magnetic monopoles condense, dual Meissner effect takes place, and (chromo)electric flux tubes end on quarks carrying (chromo)electric charge. Here we have found another description – that of magnetic dipoles, rather than monopoles, being connected by strings carrying magnetic flux. The electric charge of the fermions is screened by virtue of being in a superconductor.

Chapter 4

Spin-Flux attachment from dimensional reduction of vortices

4.1 Introduction

Physics of planar systems are of interest to physicists for nearly four decades since the discovery of integer quantum hall effect(IQHE) in 1980 by K.V. Klitzing [2]. It was shown that a 2 dimensional electron gas exposed in a high magnetic field exhibits non zero Hall conductivity at different integer filling fraction (number of magnetic flux quantum per particle). Later a even more unexpected and novel phenomenon called fractional quantum hall effect(FQHE) was seen in similar 2 dimensional semiconducting materials [3] where non-zero conductivity is seen at rational filling fraction [162]. While IQHE can be explained considering only quantized energy levels of electrons in external magnetic field called Landau levels, completely new ideas were needed for explaining FQHE. Works of Laughlin [162], Wilczek and Zee [163, 164], Jain [4] and others suggested the fact that the quasiparticles, in terms of which FQHE can be explained, are bound states of electrons and vortices (quantized flux) obeying statistics which is intermediate of bosonic or fermionic statistics. Such composite particles were called "anyons" which obey arbitrary fractional statistics. Anyons can only exist in 2+1 dimension and hence become immensely important for lower dimensional systems. Anyons and their fractional statistical property is useful not only in the context of FQHE but in the theory of parity and time reversal breaking High T_c superconductors (semionics) [7], in the context of error corrected quantum computation [19, 20, 165] etc. Recently observations of such anyonic excitations with fractional charge and statistics were reported [166, 167] in FQH type systems. These discoveries motivate further investigation into the physics of such exotic particles and systems where they can appear.

4.1.1 Overview of our work:

In Chap. 2 and in Chap. 3 we performed a dual transformation of a system consisting of a charged scalar and fermionic fields minimally coupled to electromagnetic gauge field and is expressed by the following Lagrangian

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D^{\mu}\phi^{\dagger}D_{\mu}\phi + \bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi - V(\phi^{\dagger}\phi) - eA_{\mu}\bar{\psi}\gamma^{\mu}\psi. \tag{4.1}$$

The scalar field, when condense, gives rise to vortex like collective excitations (or topological solitons). The dualization is to write the sytem in terms of new fields in which vortices and their interaction can be more explicitly understood. A duality between Abelian Higgs model and system of vortices interacting via a 2-form gauge field in 3+1 dimension was previously established. In our case due to presence of fermions a new interaction between the fermions and vortices via 2-form gauge potential appeared. The dual model is given by

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - eA_\mu\bar{\psi}\gamma^\mu\psi - \boxed{\frac{1}{2}eqB^{\mu\nu}\varepsilon_{\mu\nu\rho\lambda}\partial^\rho\frac{1}{\Box}\bar{\psi}\gamma^\lambda\psi} + \frac{1}{2}v^2\partial_\mu f\partial^\mu f$$
$$+ \frac{1}{12v^2}H^{\nu\rho\lambda}\left(\frac{1}{f^2} + \frac{q^2v^2}{\Box}\right)H_{\nu\rho\lambda} - \frac{1}{2}B_{\rho\lambda}\Sigma^{\rho\lambda} - V(f^2)\,, \tag{4.2}$$

where the boxed term is the 2-form-electron interaction which gives rise to vortex-electron interaction. We have already discussed some of it's effects in the previous chapters. In this Chapter we shall try to show how particles with fractional statistics can appear from the interaction of electrons and vortices. We know from earlier studies that particles with fractional statistics or anyons appear only in 2+1 dimensional field theories. As our original system was defined in 3+1 dimensional, to make it relevant to planar physics we need to reduce it to a 2+1 dimensional system. The novelty in our case is that such a system would give rise to anyons without presence of a Chern-Simons theory. This system was originally proposed by various authors as a phenomenological theory of High T_c superconductors. Thus this work establishes a connection of boson-fermion models with anyons as well which was previously not known.

The fact that particles in one or two spatial dimension can obey quantum statistics, other than bosonic or fermionic, was first proposed by Leinaas and Myrheim [5]. Later it was shown by Wilczek [6] that if one interchanges two composite objects, formed of electrons and vortices, their wave function picks up a phase which may not be limited to integer or half integer multiple of 2π but can be any fractional multiple of 2π. Thus the distinction between bosons and fermions, which exists in 3+1 dimension, starts to become blur in lower dimensions and there exists a continuous interpolation between these two discrete cases. Thus these composites behave as anyons. We had discussed in Chap. 1 how the concept of anyons becomes important in explanation of FQHE. As we have shown there, The picture of anyons enters the effective field theory descriptions of FQHE through Chern-Simons term. Such effective field theories first appeared in the works of Girvin and MacDonald [168], Zhang, Hansson and Kivelson (ZHK) [6] etc. A detailed discussion on this topic can be found in [65]. Let us discuss the essential features of Chern-Simons theory here onec again so that we can compare them with our work latter. The Lagrangian of a pure Chern-Simons theory is given by [56]

$$\mathcal{L} = \frac{\lambda}{2}\varepsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda - A_\mu J^\mu, \tag{4.3}$$

where A_μ is a Chern-Simons gauge field coupling to the matter current J^μ. The equation of motion of A_μ is given by

$$\varepsilon^{\mu\nu\lambda}\partial_\nu A_\lambda = \frac{1}{\lambda}J^\mu. \tag{4.4}$$

Expanding this equation we get the following two equations

$$\begin{aligned}\varepsilon_{ij}\partial_i A_j &= B = \frac{1}{\lambda}\rho,\\ \varepsilon_{ij}E_j &= \frac{1}{\lambda}J^i.\end{aligned} \tag{4.5}$$

The first equation of Eq. (4.5) expresses the fact that the magnetic field is locally proportional to charge density. Thus through Chern-Simons term flux is being bound to the charges, creating a charge-flux bound state. We shall show below how such a binding of magnetic flux to charges leads to fractional statistics and anyons. The second equation expresses that there is a current transverse to the applied electric field- a feature which characterizes Hall effect. Both of these equation are of tremendous importance in expalining the FQHE at different filling fraction.

For a non-relativistic distribution of point charges we can write

$$\rho(x) = e \sum_i \delta^2(x - x_i). \tag{4.6}$$

So the magnetic field of the Chern-Simons gauge potential is given by

$$\varepsilon_{ij}\partial_i A_j = B(x) = \frac{e}{\lambda} \sum_i \delta^2(x - x_i). \tag{4.7}$$

Thus for such a point charge distribution the magnetic field is only nonzero at points where charges exist and the value of corresponding flux is $\Phi = \frac{e}{\lambda}$. Thus each point particle will see another point particle as attached with a flux quantum $\frac{e}{\lambda}$. So when two such point charges rotate adiabetically around one another the wave function of any of the particle changes by a Aharonov-Bohm phase

$$\exp\left(ie \oint \vec{A} \cdot d\vec{x}\right) = \exp\left(i\frac{e^2}{\lambda}\right). \tag{4.8}$$

So if the particles are exchanged adiabetically the phase induced in ther wave function $2\pi\Delta\theta = \frac{e^2}{\lambda}$ which can be any arbitrary value depending on the coefficient of Chern-Simons term λ. These point particles coupled to flux quantas thus behave like anyons having fractional statistics as discussed above.

In our work we shall show that the dimentionally reduced model can also give rise to an equation similar to first equation of Eq. (4.5) expressing flux attachment and this subsequently leads to fractional statistics of flux attached fermions. For this we need to assume that vorticity has a dynamic part apart from the static part controlled by external magnetic field because not only external magnetic field but also magnetic field produced by fermionic current (including electron's spin magnetic moment) gives rise to vorticity in the condensate phase of scalar matter. Quite similar assumptions has been teken in the context of theoretical analysis of FQHE [39, 65, 169] and high temperature superconductors [170,171] where a similar gauge field is introduced via a singular gauge transformation of the nonrelativistic matter field. We have previously discussed in the Chap. 1 about the seminal work by Zhang, Hansson and Kivelson [65], where singular gauge transformation was used for statistical transmutation of a many electron problem into a bosonic problem. The bosonic theory contains a gauge field of the form $\vec{a}(x_i) = \frac{\phi_0}{2\pi}\frac{\theta}{\pi}\sum_{j\neq i}\vec{\nabla}\alpha_{ij}$, where α_{ij} is defined as the angle between the vector connecting the ith and jth particle and a reference direction. It was shown latter [39] the singular gauge transformation, in a second quantized version of the theory, leads to the Chern-Simons action for the density dependent gauge field called the statistical gauge field. Such methods are adopted in many recent works [172] as a means to achieve statistical transmutation.

The flux attachment equation derived from the dimensionally reduced theory of electron-vortex interaction is expressed below

$$\langle\epsilon_{ij}\partial_i C_j\rangle = \langle B\rangle = -\frac{\kappa e}{2m}\langle\psi^\dagger\sigma^3\psi\rangle - \frac{1}{q}\Sigma_0, \tag{4.9}$$

where $\psi^\dagger\sigma^3\psi$ is the magnetic moment density in the third direction, $(\Sigma_0)_E$ is the vorticity due to external magnetic field, and κ is a factor introduced by the process of dimensional reduction having dimension $\sqrt{M}$. We also have assumed the average charge current $\langle J_k^{NR}\rangle = 0$ in absence of external electric field. Thus the average magnetic field of C_i or vorticity is locally proportional to magnetic moment density which can also be expressed as $\langle\psi^\dagger\sigma^3\psi\rangle = \sum_i \delta^2(x - x_i)\,\delta(z - z_0)$ for discrete particles. Thus average vorticity induced by spin magnetic moment of spin polarized electrons will be localized on the particles. This seems quite similar to "flux attachment" obtained from Chern-Simons theory. This can also give rise to anyonic statistics as we shall see later in this chapter.

However in contrast to Chern-Simons gauge theory, in our case flux attachment to electrons is due to the spin of these particles. This is what is completely new in our work because there existed no mechanism or interaction which is responsible for the attachment of particles to vortices via spin magnetic moment prior to this. Now for a fully spin polarized system $\langle\psi^\dagger\sigma^3\psi\rangle \propto \rho$, the density of particles in the system, which is exactly same with Chern-Simons equation of motion. As the ZHK effective description of FQHE in terms of the Chern-Simons theory was also built upon the assumption of complete spin polarization of the electrons, this case becomes very interesting and suggest another connection of our system to the Chern-Simons gauge theory.

4.2 Dimensional reduction of the dual boson-fermion theory:

Dimensional reduction is not completely new in the context of low energy condensed matter system. One of the famous method named Kaluza-Klein reduction, although originated in the context of unification of electromagnetism and gravity, also finds application in the context low energy field theories. For example it can be applied to reduce 3+1 dimensional QED to 2+1 dimensional QED assuming a periodic structure along the extra dimension [173] which can be applied to planar systems. Recently in [174] authors used such method to derive consistent theory of time reversal invariant invariant (TRI) insulators in 3+1 dimension and in 2+1 dimensions from the theory of TRI insulators in 4+1 dimension. Also similar reduction methods are used in Schwinger particle production mechanism analysed in a planer system [175], fermionic vortex solutions [176] etc.

we consider the surface modes of electrons confined near an interface, in a small region which extends in the x-y plane and has a small thickness in the z direction. In order to do this, we take all fields to be independent of the z-coordinate, so the integration over z contributes only a length factor. The result is a dimensional reduction of the original theory. Thus in our case, the reduction is done by assuming that the fields do not depend on the third space dimension [177].

Let us start from the dual Lagrangian of boson-fermion model derived in [142],

$$\begin{aligned}\mathscr{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - eA_\mu\bar{\psi}\gamma^\mu\psi - \frac{1}{2}eMB^{\mu\nu}\varepsilon_{\mu\nu\rho\lambda}\partial^\rho\frac{1}{\Box}\bar{\psi}\gamma^\lambda\psi \\ &+\frac{1}{12}H^{\nu\rho\lambda}\left(1+\frac{M^2}{\Box}\right)H_{\nu\rho\lambda} - \frac{v}{2}B_{\rho\lambda}\Sigma^{\rho\lambda}\,.\end{aligned} \tag{4.10}$$

To reduce the system into a effective 2 dimensional system we shall use a method similar to KK method mentioned previously. In terms of Kaluza-Klein reduction method, to reduce a higher dimensional system to a lower one, one assumes one of the dimensions in the higher dimensional theory to be compactified in a circle of radius R and the radius of compactification is so small that all Kaluza-Klein modes for different fields (other than the zeroth mode) become infinitely heavy ($m_n \sim 1/R$) and thus in low energy they cannot be excited. Thus in effect propagation of modes in the higher dimension becomes frozen and only the zeroth modes confined to lower dimensional manifold becomes important. In our system no such compactified dimension is assumed. we consider only the surface modes of electrons confined near an interface, in a small region which extends in the x-y plane and has a small thickness in the z direction. In order to do this, we take all fields to be independent of the z-coordinate, so the integration over z contributes only a length factor. The result is a dimensional reduction of the original theory. Thus in our case, the reduction is done by assuming that the fields do not depend on the third space dimension [177]. This will be implemented by making derivatives in Z direction zero. For that we write the terms in dual Lagrangian separating terms containing derivative along z direction as follows.

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}F_{ab}F^{ab} - \frac{1}{2}(\partial^a A^3 - \partial^3 A^a)^2, \tag{4.11}$$

where Latin indices $a, b, ..$ represent $0, 1, 2$ coordinates. Now, according to our condition $\partial^3 A^a = 0$. Let us, for now, write $A^3 = \phi, A_3 = -\phi$ and hence we get

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}F_{ab}F^{ab} + \frac{1}{2}\partial^a\phi\partial_a\phi. \tag{4.12}$$

Similarly

$$\bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - eA_\mu\bar{\psi}\gamma^\mu\psi = \bar{\psi}(i\gamma^a\partial_a - m)\psi - eA_a\bar{\psi}\gamma^a\psi + e\phi J^3, \tag{4.13}$$

Now to reduce the terms involving Green's functions we note that when we drop the dependence of the fields on z coordinate this affectss to the Green's function as well i.e. one can write the Green's function corresponding to $\Box$ operator as

$$G(x_a - y_a, z - z') = \tilde{G}(x_a - y_a)\delta(z - z_0), \tag{4.14}$$

where z_0 denotes the z coordinate of the plane on which the fields have dynamics. Also $(\partial_0^2 - \partial_1^2 - \partial_2^2)\Delta(x_a - y_a) = \delta^3(x - y)$. Implementing this into the 2-form-fermion interaction term we have

$$\frac{1}{2}eMB^{\mu\nu}\varepsilon_{\mu\nu\rho\lambda}\partial^\rho\frac{1}{\Box}\bar{\psi}\gamma^\lambda\psi = \frac{1}{2}eMB^{ab}\varepsilon_{abc}\partial^c\frac{1}{\Delta}J^3 + eMB^a\varepsilon_{abc}\frac{1}{\Delta}\partial^b J^c, \tag{4.15}$$

where we have defined $B^a = (B^{03}, B^{13}, B^{23})$ and $\Delta = \partial_0^2 - \partial_1^2 - \partial_2^2$. Similarly, the kinetic energy term of 2-form field becomes

$$\begin{aligned}\frac{1}{12}H^{\nu\rho\lambda}\left(\frac{1}{f^2}+\frac{M^2}{\Box}\right)H_{\nu\rho\lambda} = &-\frac{1}{4}(\partial^a B^b - \partial^b B^a)^2 + \frac{1}{12}H_{abc}^2 \\ &- \frac{M^2}{4}(\partial^a B^b - \partial^b B^a)\frac{1}{\Delta}(\partial_a B_b - \partial_b B_a) + \frac{M^2}{12}H^{abc}\frac{1}{\Delta}H_{abc}\end{aligned} \quad (4.16)$$

We write the term $(\partial^a B^b - \partial^b B^a) = H^{ab}$ and the above equation as

$$\frac{1}{12}H^{\nu\rho\lambda}\left(\frac{1}{f^2}+\frac{M^2}{\Box}\right)H_{\nu\rho\lambda} = -\frac{1}{4}H_{ab}^2 + \frac{1}{12}H_{abc}^2 - \frac{M^2}{4}H^{ab}\frac{1}{\Delta}H_{ab} + \frac{M^2}{12}H^{abc}\frac{1}{\Delta}H_{abc} \quad . \quad (4.17)$$

To see what happens to vorticity $\Sigma_{\mu\nu}$, we recall its definition

$$\Sigma_{\mu\nu} = \varepsilon_{\mu\nu\rho\lambda}\partial^\rho\partial^\lambda\theta_s. \quad (4.18)$$

By writing different components of the vorticity it will be clear that only $\Sigma_{03}, \Sigma_{13}, \Sigma_{23}$ survive under our imposed condition. Hence we have

$$\frac{v}{2}B^{\mu\nu}\Sigma_{\mu\nu} = vB^a\Sigma_a + B_{ab}\Sigma^{ab}, \quad (4.19)$$

where we have written $\Sigma^a = \left(\Sigma^{03}, \Sigma^{13}, \Sigma^{23}\right)$. Thus making all fields and currents independent of third direction, we can write the new Lagrangian as

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{ab}F^{ab} + \frac{1}{2}\partial^a\phi\partial_a\phi + \bar{\psi}(i\gamma^a\partial_a - m)\psi - eA_a\bar{\psi}\gamma^a\psi + e\phi J^3 - \frac{1}{4}H_{ab}^2 + \frac{1}{12}H_{abc}^2 \\ &- \frac{M^2}{4}H^{ab}\frac{1}{\Delta}H_{ab} + \frac{M^2}{12}H^{abc}\frac{1}{\Delta}H_{abc} + \frac{1}{2}eMB^{ab}\varepsilon_{abc}\partial^c\frac{1}{\Delta}J^3 + eMB^a\varepsilon_{abc}\frac{1}{\Delta}\partial^b J^c \\ &+ vB^a\Sigma_a + B_{ab}\Sigma^{ab}.\end{aligned} \quad (4.20)$$

We note that the original action was a 4 dimensional volume integral over the Lagrangian of Eq. (4.10). After dimensional reduction, every term in Eq. (4.20) becomes independent of z direction and hence the the integral over dz contributes a length factor which has to be absorbed in redefinition of the fields. We assume that the system extends in z direction by a small length L. So all the terms in Eq. (4.20) becomes a step function along z direction i.e. they becomes constant along Z and drops to zero as $z \to \pm L$. So after integrating the 2+1 dimensional effective action would be

$$S \sim L\int d^3x\mathcal{L} \quad (4.21)$$

This factor L can be absorbed by following redefinition of fields

$$A_a \to \sqrt{L}A_a,\ \phi \to \sqrt{L}\phi,\ \bar{\psi} \to \sqrt{L}\bar{\psi},\ B_a \to \sqrt{L}B_a,\ B_{ab} \to \sqrt{L}B_{ab}. \quad (4.22)$$

This redefinition would change the Lagrangian $\mathcal{L}$, which we rename as $\mathcal{L}_R$, as follows

$$\begin{aligned}\mathcal{L}_R = &-\frac{1}{4}F_{ab}F^{ab} + \frac{1}{2}\partial^a\phi\partial_a\phi + \bar{\psi}(i\gamma^a\partial_a - m)\psi - \kappa eA_a\bar{\psi}\gamma^a\psi + \kappa e\phi J^3 - \frac{1}{4}H_{ab}^2 + \frac{1}{12}H_{abc}^2 \\ &- \frac{M^2}{4}H^{ab}\frac{1}{\Delta}H_{ab} + \frac{M^2}{12}H^{abc}\frac{1}{\Delta}H_{abc} + \frac{1}{2}\kappa eMB^{ab}\varepsilon_{abc}\partial^c\frac{1}{\Delta}J^3 + \kappa eMB^a\varepsilon_{abc}\frac{1}{\Delta}\partial^b J^c \\ &+ vB^a\Sigma_a + vB_{ab}\Sigma^{ab},\end{aligned} \quad (4.23)$$

where we have written $\frac{1}{\sqrt{L}} = \kappa$. This κ is a free parameter of our theory and will later be used in giving rise to fractional statistics of electrons. These scalings are expected on the grounds of counting mass dimensions of the fields and coupling constants. Since L is the only physical length that has appeared, it is natural that powers of L would be involved in the scaling. We also note that in the Lagrangian of Eq. (4.23) the the fields ϕ and B_{ab} couple to J_3 which is the z component of conserved fermionic current. It can be shown by Gordon decomposition that J_3 contains terms which represent currents confined to 2+1 dimensional space and having direction along z[1]. We separate terms involving $B_{a,b}$ and ϕ from rest of the Lagrangian. Thus we have

$$\mathcal{L}_R = \mathcal{L}_{2+1} + \mathcal{L}_3 \tag{4.24}$$

where $\mathcal{L}_{2+1}$ is given by

$$\begin{aligned}\mathcal{L}_{2+1} = &-\frac{1}{4}F_{ab}F^{ab} + \bar{\psi}(i\gamma^a\partial_a - m)\psi - \kappa e A_a\bar{\psi}\gamma^a\psi - \frac{1}{4}H_{ab}^2 - \frac{M^2}{4}H^{ab}\frac{1}{\Delta}H_{ab} \\ &+ \kappa e M B^a \varepsilon_{abc}\frac{1}{\Delta}\partial^b J^c + vB^a\Sigma_a,\end{aligned} \tag{4.25}$$

and the other part $\mathcal{L}_3$ is given by

$$\mathcal{L}_3 = \frac{1}{2}\partial^a\phi\partial_a\phi + \frac{1}{12}H_{abc}^2 - \frac{M^2}{12}H^{abc}\frac{1}{\Delta}H_{abc} + \kappa e\left(\phi + \frac{M}{6}\frac{1}{\Delta}\varepsilon_{abc}H^{abc}\right)J^3. \tag{4.26}$$

Thus we have got our reduced Lagrangian which will lead to flux attachment and fractional statistics of particles as we shall see in the following sections. The remaining part of the Lagrangian will not contribute to upcoming discussions and hence we shall just ignore that part for our current interest. However, we shall show later that for a static spin polarized distribution of electrons $J_3 \sim 0$ and thus two parts of the above reduced Lagrangian becomes completely decoupled. This gives a justification for our ignoring of the part $\mathcal{L}_3$.

4.3 Flux attachment to spin current:

In the theory of FQHE given by Zhang, Hansson and Kivelson [6,65] it was shown that a system containing nonrelativistic electrons in two spatial dimension may be transformed to a bosonic system by a singular transformation of many body fermionic wave function as

$$\psi \to \tilde{\psi} = \exp\left\{\left(-\sum_{i<j}\frac{\theta}{\pi}\alpha_{ij}\right)\psi\right\}. \tag{4.27}$$

For $\theta = (2n+1)\pi$ the transformed wave fuction $\tilde{\psi}$ obeys bosonic commutation rules. The transformed Hamiltonian contains a minimum coupling with the gauge field $\vec{a}$ defined as

$$\vec{a}(x_i) = \frac{\phi_0}{2\pi}\frac{\theta}{\pi}\sum_{j\neq i}\vec{\nabla}\alpha_{ij}\,, \tag{4.28}$$

[1] See Sec. 4.5 for details.

where $\phi_0 = \dfrac{hc}{e}$ is the basic unit of quantized magntic flux. In the second quantized formalism Eq. (4.28) reads

$$a_\alpha = -\frac{\phi_0}{2\pi}\frac{\theta}{\pi}\varepsilon_{\alpha\beta}\int d^2y \frac{x^\beta - y^\beta}{|\vec{x}-\vec{y}|^2}\rho(y). \tag{4.29}$$

This equation is actually a solution of the following differential equation

$$\varepsilon_{\alpha\beta}\partial^\alpha a^\beta = \frac{\theta}{\pi}\phi_0\rho, \tag{4.30}$$

and as we know this equation is actually equation of motion of Chern-Simons theory coupled to matter current. This equation suggests that in the bosonic phase for $\theta = (2n+1)\pi$ each point particle is attached to $(2n+1)$ no of flux quantum. Alternatively, one can think of each vortex as carrying a net charge of $\dfrac{e}{(2n+1)}$ i.e. individual vortices have fractional charge [178, 179]. This is called flux attachment in the context of FQHE.

To see how the condition of flux attachment can be derived from our reduced Lagrangian we shall assume that the solinoidal part of superfluid velocity around a vortex $C_\mu = \partial_\mu \theta_s$ is a dynamical variable in our theory, where besides the charged scalar, charged fermions are also present and can act as a source of magnetic field. Curl of the vector field C_μ gives the vorticity and we shall show that vorticity is non-zero where ever magnetic moment density of fermions is non-zero. As we already know that presence of external magnetic field will also create vorticity in the scalar matter, we shall only varie C_μ while keeping non-dynamical part static, the curl of which give the vorticity due to external magnetic field. So we shall write $\Sigma_\mu = \varepsilon_{\mu\nu\lambda}\partial^\nu C^\lambda + (\Sigma_\mu)_E$ later in our reduced lagrangian.

First we rewrite the reduced Lagrangian by replacing alphabets by Greek letters (in this section we shall use Latin alphabets to denote spatial indices only) as

$$\begin{aligned}\mathcal{L}_R = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - \kappa e A_\mu\bar{\psi}\gamma^\mu\psi - \frac{1}{4}H_{\mu\nu}^2 - \frac{M^2}{4}H^{\mu\nu}\frac{1}{\Delta}H_{\mu\nu} \\ &+ \kappa e M B^\mu \varepsilon_{\mu\nu\lambda}\frac{1}{\Delta}\partial^\nu J^\lambda + vB_\mu\Sigma^\mu .\end{aligned} \tag{4.31}$$

We shall now integrate out the gauge field B_μ from the Lagrangian in Eq. (4.31). For that, we separate out the relevant terms and write the partition function as

$$\mathcal{Z} = \int \mathcal{D}B_\mu \exp\left(i\int d^3x\left(-\frac{1}{4}H_{\mu\nu}^2 - \frac{M^2}{4}H^{\mu\nu}\frac{1}{\Delta}H_{\mu\nu} + \kappa e M B^\mu\varepsilon_{\mu\nu\lambda}\frac{1}{\Delta}\partial^\nu J^\lambda + vB_\mu\Sigma^\mu\right)\right). \tag{4.32}$$

We now add a gauge fixing term $\dfrac{1}{2\xi}\left(\partial_\mu B^\mu\right)^2$ to the above Lagrangian. Thus we have

$$\begin{aligned}\mathcal{Z} = &\int \mathcal{D}B_\mu \\ &\exp\left(i\int d^3x\Big(-\frac{1}{4}H_{\mu\nu}^2 - \frac{M^2}{4}H^{\mu\nu}\frac{1}{\Delta}H_{\mu\nu} + \kappa e M B^\mu\varepsilon_{\mu\nu\lambda}\frac{1}{\Delta}\partial^\nu J^\lambda + vB_\mu\Sigma^\mu - \frac{1}{2\xi}\left(\partial_\mu B^\mu\right)^2\Big)\right) \\ &= \int \mathcal{D}B_\mu \exp\left(i\int d^4x\left(\frac{1}{2}B_\mu M^{\mu\nu}B_\nu + B^\mu J_\mu^T\right)\right),\end{aligned} \tag{4.33}$$

where $M_{\mu\nu} = \left(\Delta + M^2\right) g_{\mu\nu} - \left(1 - \frac{1}{\xi} + \frac{M^2}{\Delta}\right)\partial_\mu\partial_\nu$, and $J^T_\mu = \kappa e M \epsilon_{\mu\nu\lambda}\frac{1}{\Delta}\partial^\nu J^\lambda + v\Sigma_\mu$. The Green's function corresponding to the operator $M_{\mu\nu}$ is

$$\tilde{G}_{\mu\nu} = \frac{1}{\Delta + M^2}\left(g_{\mu\nu} + \frac{1}{\Delta}\left(\xi - 1 + \frac{M^2}{\Delta}\xi\right)\partial_\mu\partial_\mu\right) . \tag{4.34}$$

So we can now perform the integral over B_μ by completing the square inside the exponent. Thus we have

$$\begin{aligned}
\mathcal{Z} &= \int \mathcal{D}B_\mu \exp\left(i\int d^3x\left(\frac{1}{2}B_\mu M^{\mu\nu}B_\nu + B^\mu J^T_\mu\right)\right)\\
&= \int \mathcal{D}B'_\mu \exp\left(i\int d^3x\left(\frac{1}{2}B'_\mu M^{\mu\nu}B'_\nu\right)\right)\exp\left(-\frac{i}{2}\int d^4x d^4y\left(J^T_\mu(x)\tilde{G}^{\mu\nu}(x-y)J^T_\nu(y)\right)\right)\\
&= N\exp\left(-\frac{i}{2}\int d^3x d^3y\left(J^T_\mu(x)\tilde{G}^{\mu\nu}(x-y)J^T_\nu(y)\right)\right)\\
&\sim \exp\Bigg(-\frac{i}{2}\int d^3x\Bigg(v^2\left(\Sigma_\mu\right)_E\frac{1}{\Delta + M^2}\left(\Sigma^\mu\right)_E + 2\kappa e v M\left(\Sigma^\mu\right)_E\frac{1}{\Delta + M^2}\tilde{J}_\mu\\
&\qquad + \kappa^2 e^2 M^2\tilde{J}_\mu\frac{1}{\Delta + M^2}\tilde{J}^\mu + 2v^2\varepsilon_{\mu\nu\lambda}\partial^\mu C^\lambda\frac{1}{\Delta + M^2}\left(\Sigma^\mu\right)_E\\
&\qquad + 2\kappa e v M C_\mu\frac{1}{\Delta + M^2}\varepsilon^{\mu\nu\lambda}\partial_\nu\tilde{J}_\lambda + \frac{v^2}{2}G_{\mu\nu}\frac{1}{\Delta + M^2}G^{\mu\nu}\Bigg)\Bigg),
\end{aligned} \tag{4.35}$$

where we have defined $G_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu$. The effective Lagrangian that we get after integrating out B_μ is

$$\begin{aligned}
\mathcal{L}_R = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - \kappa e A_\mu\bar{\psi}\gamma^\mu\psi - \frac{v^2}{2}\left(\Sigma_\mu\right)_E\frac{1}{\Delta + M^2}\left(\Sigma^\mu\right)_E\\
&- \kappa e v M\left(\Sigma^\mu\right)_E\frac{1}{\Delta + M^2}\tilde{J}_\mu - \frac{\kappa^2 e^2 M^2}{2}\tilde{J}_\mu\frac{1}{\Delta + M^2}\tilde{J}^\mu - v^2 C^\mu\frac{1}{\Delta + M^2}\varepsilon_{\mu\nu\lambda}\partial^\nu\left(\Sigma^\lambda\right)_E\\
&- \kappa e v M C_\mu\frac{1}{\Delta + M^2}\varepsilon^{\mu\nu\lambda}\partial_\nu\tilde{J}_\lambda - \frac{v^2}{4}G_{\mu\nu}\frac{1}{\Delta + M^2}G^{\mu\nu} .
\end{aligned} \tag{4.36}$$

Now what we want to do is to write an effective Lagrangian for the new emergent gauge field C_μ. To obtain that we consider the low energy limit in which $k^2 << M^2$ and the propagator can be expanded in powers of $\frac{k^2}{M^2}$. So one can get in low energy limit

$$\frac{1}{-k^2 + M^2} \sim \frac{1}{M^2}. \tag{4.37}$$

So in this low energy limit, considering only the last three terms of Eq. (4.36), we have

$$\mathcal{L}_C = -\frac{1}{q^2}C_\mu\varepsilon^{\mu\nu\lambda}\partial_\nu\left(\Sigma_\lambda\right)_E - \frac{\kappa e}{q}C_\mu\varepsilon^{\mu\nu\lambda}\partial_\nu\tilde{J}_\lambda - \frac{1}{4q^2}G_{\mu\nu}G^{\mu\nu}. \tag{4.38}$$

Now, writing $\tilde{J}_\lambda = \varepsilon_{\lambda\rho\sigma}\partial^\sigma\frac{1}{\Delta}J^\rho$ and defining C_i as $C_i \to \frac{1}{q}C_i$ we end up with the following Lagrangian for C_μ field

$$\mathcal{L}_C = -\frac{1}{4}G_{\mu\nu}G^{\mu\nu} - \frac{1}{q}C_\mu\varepsilon^{\mu\nu\lambda}\partial_\nu\left(\Sigma_\lambda\right)_E + \kappa e C_\mu J^\mu. \tag{4.39}$$

Thus we see that action of the field C_μ is similar to Maxwell's electromagnetic theory and both J_μ and $\varepsilon^{\mu\nu\lambda}\partial_\nu\left(\Sigma_\lambda\right)_E$ acts as sources. This emergent gauge theoretic structure of the vortex gauge field C_μ could have been predicted even on the basis of Eq. (3.22) in Chapter 3 given by

$$\frac{1}{q}\Sigma_{\mu\nu} = -e\varepsilon_{\mu\nu\rho\lambda}\partial^\rho\frac{1}{\Box}\bar{\psi}\gamma^\lambda\psi. \tag{4.40}$$

One can easily check that the above equation can be written as

$$\frac{1}{q}\epsilon^{\alpha\beta\mu\nu}\partial_\beta\Sigma_{\mu\nu} = -e\bar{\psi}\gamma^\alpha\psi. \tag{4.41}$$

Although the above equation seems quite unfamilier, once we write the vorticity in terms of gauge field C_μ one can see that above takes the same form as Maxwell's equation. We know that Eq. (4.40) gives rise to attachment of vorticity to spin moment density in a 3+1 dimensional case. Below we shall show that effective theory of C_μ coupled to fermions also gives rise to a similar conclusion. To proceed, we determine the equation of motion of the field C_μ

$$\partial_\mu G^{\mu\nu} = -\kappa e J^\nu + \frac{1}{q}\varepsilon^{\nu\rho\lambda}\partial_\rho\left(\Sigma_\lambda\right)_E. \tag{4.42}$$

At this point we consider time independence of sources and hence of C_0, C_i and for $\nu = i$ we can show that the equation of motion takes the following form

$$\boxed{\epsilon_{ij}\partial_i C_j = -\kappa e\tilde{J}_0 - \frac{1}{q}\left(\Sigma_0\right)_E}. \tag{4.43}$$

The equation Eq. (4.43) expresses that curl of C_i is locally prportional to $\tilde{J}_0$, which was interpreted as spin current, and to $\left(\Sigma_0\right)_E$ which is produced by external magnetic field. To see the physical meaning of this equation more closely we shall write the spin current $\tilde{J}_0$ using Gordon decomposition, ignoring interactions. Thus we write

$$\begin{aligned}\tilde{J}^0 &= \epsilon^{ik}\frac{1}{-\nabla^2}\partial_i J_k \\ &= \epsilon^{ik}\frac{1}{-\nabla^2}\partial_i\left(\frac{i}{2m}\left(\bar{\psi}\partial_k\psi - \partial_k\bar{\psi}\psi\right) + \frac{1}{m}\partial^l\left(\bar{\psi}\Sigma_{lk}\psi\right)\right).\end{aligned} \tag{4.44}$$

We recognize the first part inside parentheses as charge current $\left(J_k\right)_{cc}$ and keep it separate. Putting the identity $\Sigma_{lk} = \frac{1}{2}\varepsilon_{lk}\Sigma^3$ into Eq. (4.44) we have,

$$\begin{aligned}\tilde{J}^0 &= \epsilon^{ik}\frac{1}{-\nabla^2}\partial_i\left(\frac{i}{2m}\left(\bar{\psi}\partial_k\psi - \partial_k\bar{\psi}\psi\right) + \frac{1}{m}\partial^l\left(\bar{\psi}\Sigma_{lk}\psi\right)\right) \\ &= \frac{1}{2m}\bar{\psi}\Sigma^3\psi + \frac{1}{-\nabla^2}\varepsilon^{ik}\partial_i\left(J_k\right)_{cc} \\ &= \frac{1}{2m}\psi^\dagger\sigma^3\psi + \frac{1}{-\nabla^2}\varepsilon^{ik}\partial_i\left(J_k\right)_{cc}^{NR}\end{aligned} \tag{4.45}$$

where in the last step we have taken non relativistic approximation in which the lower component of Dirac spinor is ignored. Thus we may rewrite Eq. (4.43) using this form of $\tilde{J}^0$ as

$$\boxed{\epsilon_{ij}\partial_i C_j = -\frac{\kappa e}{2m}\psi^\dagger\sigma^3\psi + \frac{\kappa e}{\nabla^2}\varepsilon_{ik}\partial_i\left(J_k\right)_{cc}^{NR} - \frac{1}{q}\left(\Sigma_0\right)_E}. \tag{4.46}$$

The first term on the right hand side of Eq. (4.46) is the magnetic moment density in the z direction, while we can recognize $\frac{1}{-\nabla^2}(J_k)^{NR}_{cc} = A_k$ as solution of Poisons equation for $(J_k)^{NR}_{cc}$ acting as the source and hence $\varepsilon_{ik}\partial_i A_k$ is the magnetic field in the z direction produced by the electrons charge current. This equation can be compared to the Eq. (3.25) where an extra term was present which now becomes zero as we have taken all fields to be independent of z direction. This makes it possible to write the static spin moment density and the current of fermions separately. Now taking average on both side of Eq. (4.46) we get the average magnetic field of C_i

$$\langle \epsilon_{ij}\partial_i C_j \rangle = -\frac{\kappa e}{2m}\langle \psi^\dagger \sigma^3 \psi \rangle + \frac{\kappa e}{-\nabla^2}\varepsilon_{ik}\partial_i \langle (J_k)^{NR}_{cc} \rangle - \frac{1}{q}(\Sigma_0)_E \,. \tag{4.47}$$

In the absence of an external electric field the average current $\langle (J_k)^{NR}_{cc} \rangle = 0$. Also if the externally applied electric field produces current in a particular direction, the net magnetic field produced by it is zero and hence $\varepsilon_{ik}\partial_i \langle (J_k)^{NR}_{cc} \rangle = 0$. Thus in general we may write

$$\boxed{\langle \epsilon_{ij}\partial_i C_j \rangle = -\kappa\mu - \frac{1}{q}(\Sigma_0)_E \,,} \tag{4.48}$$

where $\frac{e}{2m}\langle \psi^\dagger \sigma^3 \psi \rangle = \mu$ is the local magnetic moment density at any point and for a distribution of point magnetic moments with N number of spin dipoles per unit volume we can write $\mu = \sum_{i=1}^{N} \mu_i \delta^2(\vec{x} - \vec{x_i})$, μ_i being the magnetic moment of ith particle along z direction. In our system external magnetic field can only pass in a quantized amount through the material producing static vortices. As electrons interact with vortices in such systems, there should be scattering of electrons from vortices. These scattering may give rise to an alignment of electrons along direction of magnetic field and μ become non-zero. The average value of magnetic moment per electron μ depends on strength of external magnetic fields (hence on vortex density), magnetic susceptibility of the material hosting these electrons and various other parameters.

At this point we note that Eq. (4.48) resembles the equation of motion of Chern-Simons gauge theory which expresses that vorticity is locally determined by magnetic moment density μ. This is the desired flux attachment in our case. The equation Eq. (4.48) suggests that if we take $\mu = \sum_i \mu_i \delta^2(\vec{r} - \vec{r_i})$ for discrete distribution of particles then the magnetic field of C_i at ith particle's location is $\kappa\mu_i$. We shall see in the next section that this property will lead to fractional statistics of particles.

4.4 Fractional statistics from flux attachment:

In this section we shall try to show how the attachment of flux of C_μ gauge field leads to fractional statistics of fermions minimally coupled to it. For this we shall try to derive the Schrödinger equation for the matter field from the Dirac equation by taking non-relativistic limit. We start from the reduced Lagrangian containing only the matter field coupled to C_μ.

$$\begin{aligned}\mathcal{L} =&\bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - \kappa evMC_\mu\frac{1}{\Delta+M^2}\varepsilon^{\mu\nu\lambda}\partial_\nu\tilde{J}_\lambda - v^2C_\mu\frac{1}{\Delta+M^2}\epsilon^{\mu\nu\lambda}\partial_\nu\Sigma_\lambda \\ &-\frac{v^2}{4}G_{\mu\nu}\frac{1}{\Delta+M^2}G^{\mu\nu},\end{aligned} \tag{4.49}$$

where $\tilde{J}_\lambda = \varepsilon_{\lambda\rho\sigma}\partial^\sigma\frac{1}{\Delta}J^\rho$. Putting this expression into the above Lagrangian we get

$$\begin{aligned}\mathcal{L} =&\bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + \kappa evMC_\mu\frac{1}{\Delta+M^2}J^\mu - v^2C_\mu\frac{1}{\Delta+M^2}\epsilon^{\mu\nu\lambda}\partial_\nu\Sigma_\lambda \\ &-\frac{v^2}{4}G_{\mu\nu}\frac{1}{\Delta+M^2}G^{\mu\nu}.\end{aligned} \tag{4.50}$$

Now for $k^2 << M^2$ i.e. in the low energy limit we may replace the propagrator as $\frac{1}{\Delta+M^2} \sim \frac{1}{M^2}$. Then we have

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + \kappa eC_\mu J^\mu - \frac{1}{q}C_\mu\epsilon^{\mu\nu\lambda}\partial_\nu\Sigma_\lambda - \frac{1}{4}G_{\mu\nu}G^{\mu\nu}, \tag{4.51}$$

where we have redefined C_μ as $\frac{1}{q}C_\mu$. we can see that in the low energy limit C_μ is minimally coupled to fermions. Let us now derive the Dirac equation from the above and is given by

$$(i\gamma^\mu\partial_\mu - m + \kappa e\gamma^\mu C_\mu)\psi = 0. \tag{4.52}$$

To reduce this equation to a Schrödinger equation we first note that fermionic matter acts as a source of field C_μ. For a nearly static fermion the C_μ field will also be time independent and as the fermion has a non zero average magnetic moment, $\langle\vec{\nabla}\times\vec{C}\rangle$ is non zero only at particle's location. So if a second flux attached fermion is taken around the first one, it sees a time independent configuration of C_μ with a net zero $\langle\vec{\nabla}\times\vec{C}\rangle$ at its position. We now want to derive Schrödinger equation for the second fermion moving around first one.

For that we take the time dependence of the fermionic field to be $\psi(\vec{x},t) = \psi(\vec{x})e^{(-iEt)}$. Putting this into the above Dirac equation we can derive the following two equations

$$\begin{aligned}\sigma^i(\hat{p}_i + \kappa eC_i)\psi_B &= -(E + \kappa eC_0 - m)\psi_A, \\ \sigma^i(\hat{p}_i + \kappa eC_i)\psi_A &= -(E + \kappa eC_0 + m)\psi_B\end{aligned} \tag{4.53}$$

Now we take the approximation that in nonrelativistic limit $E \simeq E_{NR} + m$. Thus

$$\begin{aligned}\psi_B &= -\frac{1}{(E+\kappa eC_0+m)}\sigma^i(\hat{p}_i - \kappa eC_i)\psi_A \\ &\simeq \frac{1}{(E_{NR}+\kappa eC_0+2m)}\sigma^i(\hat{p}_i - \kappa eC_i)\psi_A \\ &= \frac{1}{2m}\left(1 - \frac{(E_{NR}+\kappa eC_0)}{2m} + ...\right)\sigma^i(\hat{p}_i - \kappa eC_i)\psi_A,\end{aligned} \tag{4.54}$$

where we have assumed that $(E_{NR} + \kappa eC_0) << 2m$. Now we take only first term of the expansion and put it into first equation of (4.53). After some algebraic manipulation we can reach our desired Schrödinger equation

$$\left[-\frac{1}{2m}\left(\vec{\nabla} + i\kappa e\vec{C}\right)^2 + \kappa e\sigma^3(\vec{\nabla}\times\vec{C})_3 + \kappa eC_0\right]\psi_A = E_{NR}\psi_A. \tag{4.55}$$

As discussed previously if the second electron goes in a closed path around the first nearly static electron then on the path $\langle \vec{\nabla} \times \vec{C} \rangle = 0$. Hence for the second electron we have

$$\left[-\frac{1}{2m} \left(\vec{\nabla} + i\kappa e \vec{C} \right)^2 + \kappa e C_0 \right] \psi = E_{NR} \psi. \tag{4.56}$$

So for a closed path Γ around the static electron the solution of the above Schrödinger equation may be written as

$$\begin{aligned} \psi(x) &= \psi_0(x) \exp \left(-i\kappa e \oint_\Gamma \vec{C} \cdot \vec{dl} \right) \\ &= \psi_0(x) \exp \left(-i\kappa e \int_s \left(\vec{\nabla} \times \vec{C} \right) \cdot \vec{ds} \right). \end{aligned} \tag{4.57}$$

Now in a distribution of spin polarized fermions for which the average spin magnetic moment per electron is μ_0, the above solution gives

$$\psi_0(x) = \psi_0(x) \exp \left(ie\kappa^2 \mu_0 \right), \tag{4.58}$$

where it was also assumed that the closed path of the 2nd fermion encloses no externally created vortex. The above equation says that for spin polarized and nearly static distribution of electron the change in phase of the second particle going around the first in a closed path Γ is $\frac{e\mu_0}{L}$, where we have written $\kappa^2 = \frac{1}{L}$. Thus for a non zero average magnetic moment per particle there is a extra phase change in interchanging two particle which is

$$\Delta\theta = \frac{e\mu_0}{L}. \tag{4.59}$$

As $\mu_0 \sim \frac{e}{2m}$ the phase will be of the order of $\frac{e^2}{mL}$.Thus particles attached to fluxes of C_μ obeys fractional statistics i.e. they become anyons due to arbitrariness of L. At this point we note that the corresponding phase factor in Chern-Simons theory is $\frac{e^2}{\lambda}$, λ being the coefficient of the Chern-Simons term in action. Thus the factor which is comparable to λ in our theory is mL which is a dimension less contant.

4.5 Discussions

Usually an investigation of 2+1-dimensional physics starts from a theory defined on a plane. In this paper, we have attempted to describe 2+1 dimensional phenomena by starting in 3+1 dimensions, from the dual of a theory of magnetic flux strings coupled to electrons, and reducing it to one dimension less. The higher dimensional theory shows a linear potential between pairs of electrons, which can be interpreted as the result of a flux string connecting the spin magnetic moments of the two electrons. When reduced dimensionally by freezing all motion in one direction, the strings are reduced to vortices and the magnetic flux at each particle's location is proportional to the magnetic moment of the particle. This indicates a "flux attachment" similar to that arising in the effective description of fractional quantum Hall effect through the Chern-Simons term. We also find that similar to the FQHE description, the particles with attached flux exhibit fractional statistics. In this chapter we have also found that spin magnetic moment is the central cause of attachment of flux to particles. This is quite natural because vorticity will appear wherever magnetic field penetrates the superconducting bulk. Thus in our framework such flux attachment or flux pinning would be impossible for spinless (scalar) charged particles, which is in stark contrast with similar phenomena in Chern-Simons theories.

A brief comparison with Chern-Simons theory may be relevant here. It is well known that the Chern-Simons gauge field has no dynamics of its own when it is not coupled to matter. When a coupling to matter current is added, it constrains the magnetic field to be nonvanishing only at the locations of the charged particles. The electric field in any direction also turns out to be proportional to the transverse current, which is a characteristic feature of Hall effect. We have started from the Abelian Higgs system in the broken phase, with flux strings as well as electrons, and dualized it to a $B \wedge F$-type theory. Unlike pure Chern-simons gauge theory, it is not a purely topological field theory, but when reduced to 2+1 dimensions, the coupling between the vortices and electrons also constrains the magnetic field to be nonvanishing only at the locations of the particles. This phenomenon of flux attachment leads to fractional statistics of the particles just as for Chern-Simons theory, but it will require further investigations to check if Hall effect is also present in the setup that we have considered.

The attachment of magnetic flux to spin (or more precisely magnetic moment) that we have found above may be compared with the ad hoc construction adopted in [180], where electrons with up and down spins were treated as bosonic operators interacting with a Chern-Simons gauge field which attaches flux to spin. Then in the bosonic variables, a dual theory of vortices could be constructed and different phases could be analyzed. We have shown that flux attachment to spin can occur in a specific type of boson-fermion model, namely the Abelian Higgs model with itinerant electrons; we can expect that similar phases, e.g. chiral spin liquid, or superconductors with tightly bound pairs, will be realized in this system as well.

Finally, we would like to comment on the $\mathscr{L}_3$ which we ignored after Sec. 4.2. This is justified in the lowest order, with the fermions being treated as noninteracting, so that J_3 is independent of the other two components of the fermion current. In general, interactions will ensure that the effects of $\mathscr{L}_3$ should be taken into account. But there is a special situation when $\mathscr{L}_3$ decouples from $\mathscr{L}_{2+1}$ – when there is complete spin polarization of electrons along the external magnetic field. To see this we write J_3 by using Gordon's decomposition,

$$J^3 = \frac{1}{m}\partial_\nu(\bar{\psi}\Sigma^{3\nu}\psi) = \frac{1}{m}\partial_0(\bar{\psi}\Sigma^{30}\psi) + \frac{1}{m}\partial_i(\bar{\psi}\Sigma^{3i}\psi)\,. \tag{4.60}$$

We assume the fermionic fields to be slowly varying in time. Then the first term is small. The second term is proportional to $\varepsilon^{ij}\partial^i(\psi^\dagger\sigma^j\psi)$, where ψ is the nonrelativistic two-component Pauli spinor. Thus in a fully spin polarized state of fermions, $J^3 \approx 0$ and the two parts decouple from each other. Another important feature of this special state is that in this case the flux attachment equation takes the form

$$\langle\epsilon_{ij}\partial_i C_j\rangle = -\frac{\kappa e}{2m}(\rho_\uparrow - \rho_\downarrow) = -\frac{\kappa e}{2m}\rho\,, \tag{4.61}$$

where $\rho = \rho_\uparrow$ is the density of particles. Thus we see that in this case the flux attachment equation reduces to the form obtained in Chern-Simons theory. These observations make the spin polarized system very special and demands further investigation.

Chapter 5

Discussions and Future Directions

In this thesis we have mainly investigated into the dual formulation of an Abelian Higgs model coupled to charged fermions and some consequences of the dual model. In particular we have shown how a $B \wedge F$ term may induce in the effective low energy theory if fermions are integrated out from the theory. Induction of $B \wedge F$ term due to fermionic loop correction is rare and only a few example exists [59, 181]. Thus the boson fermion model becomes another example which in dual description allows the induction of $B \wedge F$ term via fermionic loop correction. At this point we would like to mention that $B \wedge F$ term is considered as the effective topological field theory for bosonic topilogical insulators (BTI) [54, 130, 182]. Thus the dual model may hold properties of a BTI. But this direction is not investigated in our thesis and may be set as one of the future research goal.

We have also shown that how in the dual theory the phenomenon of spin-vorticity attachment would appear and lead to linear interaction between fermions for exchange momentum $k^2 >> M^2$. The explation for this can be given in terms of spin-vorticity attachment as described in Chap. 3. We also have shown that the dual theory of vortex lines or strings can be generalized in terms of a string field theory. The false vacuum of the this string field description may give rise to linearly rising potential between fermions. Such a potential along with Coulombic repulsion gives rise to the fermion pairs. It is well known that formation of fermionic pairs is necessary for appearance of superconductivity. Thus one may investigate into the nature of superconductivity arising from these fermion pairs found in our work. We had previously discussed about the connection of boson fermion models to high T_c superconductors in Chap. 2. Pseudogap plays an important role in the phenomenology of cuprate superconductors. As the fermion pairs in our dual theory appears for higher exchange momentum (hence at higher energy scale) we may expect such a pairing to contribute in pseudogap phenomenology. Hence this also becomes a possible future direction.

This attachment of vortex to electrons looks quite like the flux tube model of anyons proposed by Wilczek [6]. As we had discussed previously in Chap. 4, anyons are particles which exists in two spatial dimension having fractional statistical property as well as fractional charge. To know whether the vortex attached fermions obey such properties we first reduce our theory from 3+1 dimension to 2+1 dimension. The dimemsionally reduced theory also gives rise to the vortex-spin attachment. We have also shown that a new parameter of dimension $\sqrt{M}$ is induced in the 2+1 dimensional theory. If this parameter can be tuned sufficiently then the vortex-spin attachment will lead to anyonic phase factor. However, we have not yet shown the property of fractional charge for these vortex-electron composites. Thus it remains an open question. Also, in our case no Chern-Simons term is used to arrive at the flux attachment condition. It is well known that the anyons that appear in the Chern-Simons gauge theory breaks time reversal(T) and parity(P) symmetry. Whether such a symmetry breaking occurs in our case also remains an open question as well.

The system we have considered consists of bosonic and fermionic fields. The bosonic field in the symmetry broken state represent a superconductor. Thus realization of such physics can be found if one constructs a hybrid system which consists of one superconducting block in close proximity with another block which host a gas of electrons. Such materials could be Graphene, GAAlAs-GaAs heterostructure, or topological insulator (TI) which hosts relativistic or non-relativistic electrons. Now, if one passes magnetic field through the hybrid system it will create vortices as well as spin polatized electrons. A schamatic picture of such a system is shown below in Fig. 5.1

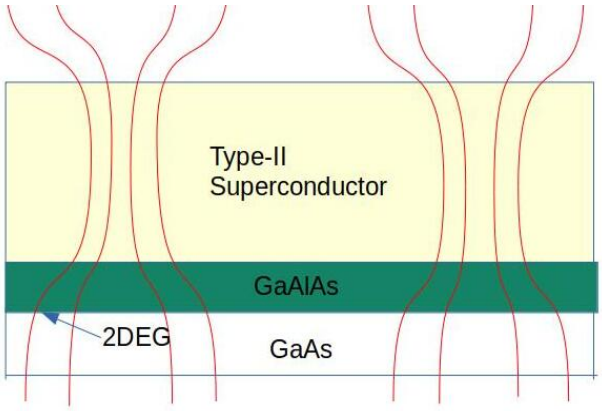

Figure 5.1: Vortex passing through Superconductor-2DEG Hybrid

As vortices and electrons are in close proximity in such systems, it is possible that they start to interact in the way we have described in this thesis and the phenomenon of flux attachment is realized.

Previously such GAAlAs-GaAs heterostructure- Superconductor hybrid was used to study the Hall effect in an inhomogeneous magnetic field produced in a distribution of vortices [183–185]. Some non-trivial deviation from classical Hall effect was found out in these studies. As in 2+1 dimensional dual theory vortices are represented by point particles one can calculate the scattering of electrons from point vortices in a field theoretic manner by writing down a field theory for the point like vortices. One can calculate the interaction potential between vortices and electrons from the scattering amplitude. With the help of this potential one can calculate transport properties of electrons in a distribution of vortices and compare with the existing results. In this way one can check the validity of this theoretical duality as well as find out some new features about the transport of electrons in presence of vortices.

Lastly, the boson-fermion model with which we were working with had only electromagnetic interactions in it. However, in all the model systems where such a boson fermion mixture was thought to arise, a direct interaction between fermions and bosons was also assumed to take place. In all such cases the form of the interaction that was considered is shown below

$$\mathcal{L}_I \sim \phi^\dagger \psi_\uparrow \psi_\downarrow + \phi \psi_\downarrow^\dagger \psi_\uparrow^\dagger. \tag{5.1}$$

In relativistic models the form of the interaction that was taken looks like

$$\mathcal{L}_I \sim i\frac{g^*}{2}\phi^*\bar{\psi}^c\psi - i\frac{g}{2}\phi\bar{\psi}\psi^c, \tag{5.2}$$

where ψ_c represents the charge conjugated form of fermionic fields. One can try to dualize the boson-fermion model in presence of such interaction terms to give realistic predictions based on the dual theory. In particular such dual theory may help us in understanding different interesting features about topological insulator (TI)-Superconductor surface. Interface of such jucntions is known to show interesting physical properties. It was previously shown that a vortex solution in the superconductor side of such a junction has a fractional electric charge as well as fractional angular momentum due to presence of the axion term at the boundary [186]. Another interesting feature of such junctions is that, due to the proximity effect, expressed by the cubic coupling terms shown above, the Hamiltonian of the surface modes can be projected as that of a $p_x + ip_y$ type superconductor [18]. The vortices in $p_x + ip_y$ superconductor are known to hold Majorana zero modes (MZM) [28]. As the dualization expresses vortices and their interactions more explicitly in the dual model, it may lead to deep insights into these phenomena as well.

Appendices

.1 Villain Approximation:

Here we shall try to approach Villain approximation in the context of 2 dimensional X-Y model in both physical and mathematical point of view. Let us start from the physical point of view first. The partition function of the 2 dimensional X-Y model is given by

$$\mathcal{Z} = \int \prod_n d\theta_n \exp\left(-\beta J \sum_{\langle r,r'\rangle} S_r \cdot S_{r'}\right) = \int \prod_n d\theta_n \exp\left(-\beta \sum_{n\mu} \cos\left(\theta_{n+\hat{\mu}} - \theta_n\right)\right), \quad (3)$$

where the spin variables S_i have continuous rotation and thus hold $U(1)$ symmetry. Let us first assume that J, the nearest neighbour coupling strength, is positive. Let us recall the fact that the partition function Z represents sum of probabilities of all possible configuration with different values $\theta_{n+\mu}$ at all possible lattice points n and associated links μ. If now, we start to decrease temperature i.e. take the limit $\beta \to \infty$ the exponent would start to decrease rapidly and to keep it non zero the $\cos\theta$ approches its maximum i.e. $\theta \to 0$. This would mean that at low temperature all spins would start to get ordered such that $\Delta_\mu \theta_n \to 0$. Again, as θ is a periodic variable θ the exponential function must remain unchanged under the substitution $\theta \to \theta + 2m\pi$, where m is an integer variable. Thus in the large β limit one can do the following replacement

$$\exp\left(-\beta \cos\left(\theta_{n+\hat{\mu}} - \theta_n\right)\right) \to \sum_{m=-\infty}^{\infty} e^{-\beta} \exp\left(\frac{\beta}{2}\left(\theta_{n+\hat{\mu}} - \theta_n + 2\pi m\right)^2\right), \quad (4)$$

where we have expanded $\cos\left(\Delta\theta - 2\pi m\right)$ in small $(\Delta\theta - 2\pi m)$ limit. This approximation given by Eq. (3) is called "Villain approximation".

Let us now try to see another method of obtaining same approximation. We again write down the hamiltonian of of 2 dimensional X-Y model as

$$H = J\cos\left(\theta_{n+\hat{\mu}} - \theta_n\right). \quad (5)$$

We know that a periodic like this can always be written in terms of Fourier series as

$$e^{-\beta H(\theta)} = \sum_{s=-\infty}^{\infty} e^{is\theta + \tilde{H}(\theta)}, \quad (6)$$

where the Fourier can be written as

$$e^{\tilde{H}(\theta)} = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-is\theta - \beta H(\theta)}. \quad (7)$$

We are interested in the large β or low temperature limit. To derive the approximate form of $e^{-\beta H(\theta)}$ in this limit we use Poisson's summation formula

$$\sum_{s=-\infty}^{\infty} g(s) = \sum_{m=-\infty}^{m=\infty} \int_{-\infty}^{\infty} d\phi g(\phi) e^{-i2m\pi\phi}. \quad (8)$$

Using $g(s) = e^{is\theta + \tilde{H}(\theta)}$ we can get

$$e^{\beta H(\theta)} = \sum_{m=-\infty}^{+\infty} \int d\phi e^{i(\theta - 2\pi m)\phi} e^{\tilde{H}(\phi)} = \sum_{m=-\infty}^{+\infty} e^{H_0(\theta - 2\pi m)}, \quad (9)$$

where we have written

$$e^{H_0(\theta - 2\pi m)} = \int d\phi e^{i(\theta - 2\pi m)\phi} e^{\tilde{H}(\phi)}. \quad (10)$$

In the large β limit one can take the choice $H_0(\theta - 2\pi m) \simeq -\frac{1}{2}K(\theta - 2\pi m)^2$, where K is a new constant. Using this we land up again on Villain approximation [122, 187]

$$e^{\beta H(\theta)} \simeq \sum_{m=-\infty}^{+\infty} e^{-\frac{1}{2}K(\theta-2\pi m)^2}. \tag{11}$$

.2 Vacuum Polarization:

Here we shall show the calculation of vacuum polarization diagram, with 2-form-femion interaction present, in the path integral method. Let us start from the 3+1 dimensional dual Lagrangian as

$$\begin{aligned}\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - eA_\mu\bar{\psi}\gamma^\mu\psi - \frac{1}{2}eqB^{\mu\nu}\varepsilon_{\mu\nu\rho\lambda}\partial^\rho\frac{1}{\Box}\bar{\psi}\gamma^\lambda\psi + \frac{1}{2}v^2\partial_\mu f\partial^\mu f \\ + \frac{1}{12v^2}H^{\nu\rho\lambda}\left(\frac{1}{f^2} + \frac{q^2v^2}{\Box}\right)H_{\nu\rho\lambda} - \frac{1}{2}B_{\rho\lambda}\Sigma^{\rho\lambda} - V(f^2)\,.\end{aligned} \tag{12}$$

To proceed let us separate out the part of the Lagrangian containing fermionic fields

$$\mathcal{L}_{\psi,\bar{\psi}} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - eA_\mu\bar{\psi}\gamma^\mu\psi - \frac{1}{2}MqB^{\mu\nu}\varepsilon_{\mu\nu\rho\lambda}\partial^\rho\frac{1}{\Box}\bar{\psi}\gamma^\lambda\psi\,, \tag{13}$$

where we have rescaled the 2-form field as $B_{\mu\nu} \to vB_{\mu\nu}$. Next we perform an integration by parts in the third term to write it as

$$\int d^4x\,\frac{1}{2}MqB^{\mu\nu}\varepsilon_{\mu\nu\rho\lambda}\partial^\rho\frac{1}{\Box}\bar{\psi}\gamma^\lambda\psi = \int d^4x\,\frac{1}{2}\frac{eM}{\Box}\varepsilon_{\mu\nu\rho\lambda}\partial^\nu B^{\rho\lambda}\bar{\psi}\gamma^\mu\psi. \tag{14}$$

Next let us define $A^{\text{eff}}_\mu = A_\mu + M\Box^{-1}F_\mu$ where we have written $F_\mu = \frac{1}{2}\varepsilon_{\mu\nu\rho\lambda}\partial^\nu B^{\rho\lambda}$. With this the Lagrangian takes the following form

$$\mathcal{L}_{\psi,\bar{\psi}} = \bar{\psi}(i\gamma^\mu\partial_\mu - e\gamma^\mu A^{\text{eff}}_\mu - m)\psi. \tag{15}$$

We shall now integrate out fermionic fields by writing the generating functional as

$$\mathcal{Z} = \int \mathscr{D}\bar{\psi}\mathscr{D}\psi\,\exp\left(i\int d^4x\Big(\bar{\psi}(i\gamma^\mu\partial_\mu - e\gamma^\mu A^{\text{eff}}_\mu - m)\psi\Big)\right), \tag{16}$$

We shall expand the action around the stationary action containing fermionic fielfds which satisfy the Euler-Lagrange equation as follows

$$\begin{aligned}S =& S|_{\psi_0,\bar{\psi}_0} + \int d^4x\frac{\delta S}{\delta\psi}\bigg|_{\psi_0,\bar{\psi}_0}(\psi-\psi_0) + \int d^4x(\bar{\psi}-\bar{\psi}_0)\frac{\delta S}{\delta\bar{\psi}}\bigg|_{\psi_0,\bar{\psi}_0} \\ &+ \frac{1}{2!}\int d^4x d^4y(\bar{\psi}-\bar{\psi}_0)(x)\frac{\delta^2 S}{\delta\bar{\psi}\delta\psi}\bigg|_{\psi_0,\bar{\psi}_0}(\psi-\psi_0)(y) + \ldots.\,,\end{aligned} \tag{17}$$

Now at $\psi = \psi_0$, $\bar{\psi} = \bar{\psi}_0$ the second and third terms of the above action are zero as at this point the action is stationary. Also we define $(\psi - \psi_0) = \eta, (\bar{\psi} - \bar{\psi}_0) = \bar{\eta}$ and rewrite the generating functional as

$$Z = \int \mathcal{D}\eta\mathcal{D}\bar{\eta}\exp\left[i\int d^4x\Big(S|_{\psi_0,\bar{\psi}_0} + \frac{1}{2!}\int d^4y\ \bar{\eta}(x)(i\gamma^\mu\partial_\mu - e\gamma^\mu A^{\text{eff}}_\mu - m)\eta(y)\Big)\right], \tag{18}$$

where we have neglected terms higher than square terms and have written

$$\left.\frac{\delta^2 S}{\delta\bar{\psi}\delta\psi}\right|_{\psi_0,\bar{\psi}_0} = (i\gamma^\mu\partial_\mu - e\gamma^\mu A^{\text{eff}}_\mu - m). \tag{19}$$

Integrating out fermionic fields $\eta,\ \bar{\eta}$ from Eq. (18) one gets the following expression

$$Z = e^{iS|_{\psi_0,\bar{\psi}_0}} \quad det\Big[i\left(i\gamma^\mu\partial_\mu - m - e\gamma^\mu A^{\text{eff}}_\mu\right)\Big]. \tag{20}$$

This expression of the generating functional contains all the quantum effect of fermions in terms of the gauge fields A_μ and $B_{\mu\nu}$. To determine the quantum effects upto one fermion loop one need to expand the determinant in the following way.

$$\begin{aligned}
&\left(i\gamma^\mu\partial_\mu - m - e\gamma^\mu A^{\text{eff}}_\mu\right)\\
&= (i\gamma^\mu\partial_\mu - m)\left(1 - \frac{1}{(i\gamma^\mu\partial_\mu - m)}e\gamma^\mu A^{\text{eff}}_\mu\right)\\
&= (i\gamma^\mu\partial_\mu - m)\exp\mathrm{Tr}\left[\ln\left(1 - \frac{1}{(i\gamma^\mu\partial_\mu - m)}e\gamma^\mu A^{\text{eff}}_\mu\right)\right]\\
&= (i\gamma^\mu\partial_\mu - m)\exp\left[-\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{Tr}\left(\frac{1}{(i\gamma^\mu\partial_\mu - m)}e\gamma^\mu A^{\text{eff}}_\mu\right)^n\right].
\end{aligned} \tag{21}$$

The first term in this expansion can be re written as

$$\mathrm{Tr}\left(\frac{1}{(i\gamma^\mu\partial_\mu - m)}e\gamma^\mu A^{\text{eff}}_\mu\right) = \mathrm{Tr}\left[\frac{i\gamma^\mu\partial_\mu - m}{\Box + m^2}e\gamma^\mu A^{\text{eff}}_\mu\right]. \tag{22}$$

This term is called a tadpole term and gives null contribution. This can be most easily seen by taking Fourier transform of the argument of Tr. In momentum space this expression read

$$\mathrm{Tr}\left[\int\frac{d^4k}{(2\pi)^2}\frac{i\gamma^\mu k_\mu - m}{k^2 - m^2}e\gamma^\mu A^{\text{eff}}_\mu(k)\right]. \tag{23}$$

We can easily see that the first term of numerator would be zero as it is odd in K and second term would be zero as $\mathrm{Tr}(\gamma^\mu) = 0$. Let us now proceed towards evaluating the second term in the expansion in Eq. (21). Taking a Fourier transform of this term we shall land up on

$$-\frac{1}{2}\mathrm{Tr}\left[e^2\int\frac{d^4p}{(2\pi)^2}\frac{d^4q}{(2\pi)^2}\frac{((q+p)_\mu\gamma^\mu + m)\,\gamma^\alpha A^{\text{eff}}_\alpha\,(q_\nu\gamma^\nu + m)\,\gamma^\beta A^{\text{eff}}_\beta}{[(q+p)^2 - m^2]\,[q^2 - m^2]}\right]. \tag{24}$$

To evaluate this expression we modify the denominator by introducing Feynman parameter z as follows

$$\frac{1}{[(q+p)^2 - m^2]\,[q^2 - m^2]} = \int_0^1\frac{dz}{\{z\,[(q+p)^2 - m^2] + (1-z)\,[q^2 - m^2]\}^2}. \tag{25}$$

Evaluating the the denominator, we can rewrite Eq. (25) as

$$\int_0^1\frac{dz}{\{z\,[(q+p)^2 - m^2] + (1-z)\,[q^2 - m^2]\}^2} = \int_0^1\frac{dz}{[(q+pz)^2 + p^2 z(1-z) - m^2]^2}. \tag{26}$$

Let us now rename $(q + pz) = l$ and then change variables from q to l. Diue to this change of variable there will be terms odd in l in the numerator. These terms will vanish and we shall get the following expression for Eq. (24)

$$
\begin{aligned}
&-\frac{1}{2}e^2 \int \frac{d^4p}{(2\pi)^4} A^{\text{eff}}_\alpha(p) A^{\text{eff}}_\beta(-p) \\
&\int \frac{d^4l}{(2\pi)^4} \int_0^1 dz \frac{4\left[2l^\alpha l^\beta - 2p^\alpha p^\beta x(1-x) - l^2 g^{\alpha\beta} + p^\mu p_\mu z(1-z) g^{\alpha\beta} + m^2 g^{\alpha\beta}\right]}{\left[l^2 + Q^2\right]^2}, .
\end{aligned}
\tag{27}
$$

where we have written $Q^2 = p^2 z(1-z) - m^2$ and used the formulas $\text{Tr}\{\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\} = 4\left(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta} + g^{\mu\beta}g^{\nu\alpha}\right)$ and $\text{Tr}\left\{\gamma^\alpha\gamma^\beta\right\} = 4g^{\alpha\beta}$. To proceed further, we now rewrite the numerator of Eq. (27) as

$$
\begin{aligned}
&\left[2l^\alpha l^\beta - 2p^\alpha p^\beta z(1-z) - l^2 g^{\alpha\beta} + p^\mu p_\mu z(1-z) g^{\alpha\beta} + m^2 g^{\alpha\beta}\right] \\
&= \left[2l^\alpha l^\beta + 2z(1-z)\left(p^2 g^{\alpha\beta} - p^\alpha p^\beta\right) - \left(l^2 + Q^2\right) g^{\alpha\beta}\right].
\end{aligned}
\tag{28}
$$

Thus the integral over l can be divided into three different parts as follows

$$
\int \frac{d^4l}{(2\pi)^4} \left[\frac{2l^\alpha l^\beta}{\left[l^2 + Q^2\right]^2} + \frac{2z(1-z)\left(p^2 g^{\alpha\beta} - p^\alpha p^\beta\right)}{\left[l^2 + Q^2\right]^2} - \frac{g^{\alpha\beta}}{\left[l^2 + Q^2\right]}\right].
\tag{29}
$$

The first and last term of Eq. (29) will exactly cancell each other in the large Λ limit, while the second term will remain and give the only contribution to our calculation. To evaluate this term we shall rotate l_0, p_0 in the complex plane to make them $l_0^E = il_0$, $p_0^E = ip_0$. Under such transformation we shall have $l^2 = -l_E^2$, $d^4l = id^4l_E$, $d^4p_E = id^4p_E$. Also one can show that

$$
A^{\text{eff}}_\alpha(p)\left(p^2 g^{\alpha\beta} - p^\alpha p^\beta\right) A^{\text{eff}}_\beta(-p) = A^{\text{eff}}_{E\alpha}(p)\left(p_E^2 g^{\alpha\beta} - p_E^\alpha p_E^\beta\right) A^{\text{eff}}_{E\beta}(-p),
\tag{30}
$$

where subscript E variables in right hand side are the Euclideanized variables. With all this modification we can evaluate Eq. (27) as

$$
\begin{aligned}
&-\frac{1}{2}e^2 \int \frac{id^4p_E}{(2\pi)^4} A^{\text{eff}}_{E\alpha}(p_E) A^{\text{eff}}_{E\beta}(-p_E) \\
&\int_0^1 dz z(1-z)\left(p_E^2 g^{\alpha\beta} - p_E^\alpha p_E^\beta\right) \int \frac{id^4l_E}{(2\pi)^4} \frac{1}{\left[l_E^2 + Q_E^2\right]^2},
\end{aligned}
\tag{31}
$$

This Euclidean integral can be evaluaterd very easily witha cutoff momentum Λ and the result is given by

$$
-\frac{i}{2}e^2 \int \frac{d^4p}{(2\pi)^4} A^{\text{eff}}_\alpha(p)\pi(p^2)\left(p^2 g^{\alpha\beta} - p^\alpha p^\beta\right) A^{\text{eff}}_\beta(-p)\,,
\tag{32}
$$

where we have written

$$
\pi(p^2) = \int_0^1 dz z(1-z)\left[\ln\left(1 + \frac{\Lambda^2}{m^2 - p^2 x(1-x)}\right) - \frac{\Lambda^2}{\Lambda^2 + m^2 - x(1-x)p^2}\right].
\tag{33}
$$

Also we have rotated Euclidean variables to normal ones. This equation Eq. (32) is the desired expression for the vacuum polarization in momentum space. One can take a furier transform of this to obtain the expression in coordinate space.

.3 Dualization of 2+1 D boson-fermion model:

Here we shall discuss the dualization of 2+1 dimensional boson-fermion model. The model of complex scalar field coupled to massive Dirac fermions in 2+1 dimension via electromagnetic gauge field is expressed by

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D^{\mu}\phi^{\dagger}D_{\mu}\phi + \bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi - V(\phi^{\dagger}\phi) - eA_{\mu}\bar{\psi}\gamma^{\mu}\psi\,, \tag{34}$$

where now the Dirac fermions are defined in 2+1 dimension and the gamma matrices are expressed by Pauli spin matrices. One set of choice of gamma matrices are

$$\gamma^0 = \sigma^3, \quad \gamma^1 = i\sigma^1, \quad , \gamma^2 = \sigma^2. \tag{35}$$

To proceed we take methods exactly similar to that shown previously in this thesis. The complex scalar field ϕ can be expressed in polar form as $\phi = \rho e^{i\theta}$. With this we can write the scalar field lagrangian as

$$|D_{\mu}\phi|^2 = (\partial_{\mu}\rho)^2 + \rho^2(\partial_{\mu}\theta + qA_{\mu})^2. \tag{36}$$

This would lead to the following form of the Lagrangian Eq. (34)

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\partial_{\mu}\rho)^2 + \rho^2(\partial_{\mu}\theta + qA_{\mu})^2 + \bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi \\ &- V(\rho) - eA_{\mu}\bar{\psi}\gamma^{\mu}\psi\,,\end{aligned} \tag{37}$$

As we have seen before, to bring vortices into the picture we decompose the phase θ of the scalar field as $\theta = \theta_s + \theta_r$, where θ_r represents the regular valued fluctuation of the phase which actually represents the irrotational part of the superfluid velocity. With this decomposition we can proceed to linearize the third term in the Lagrangian Eq. (37) by introducing into the generating functional a Gaussian term like

$$\int \mathcal{D}C_{\mu} e^{-\frac{i}{2}\int d^3x[C_{\mu}+\rho(\partial_{\mu}\theta+qA_{\mu})]^2} = \mathcal{N}, \quad A\ constant. \tag{38}$$

The linearized Lagrangian is given by

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\partial_{\mu}\rho)^2 - \frac{1}{2}C_{\mu}C^{\mu} - \rho C_{\mu}(\partial^{\mu}\theta^r + \partial^{\mu}\theta^s) - q\rho C_{\mu}A^{\mu} + \bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi \\ &- V(\rho) - eA_{\mu}\bar{\psi}\gamma^{\mu}\psi\,,\end{aligned} \tag{39}$$

If we now integrate out the auxiliary field θ^r it will give us a delta function constraint which imposes constraint over the newly introduced decoupling field C_{μ}. This is given by

$$\partial^{\mu}(\rho C_{\mu}) = 0. \tag{40}$$

This can be satisfied in 2+1 dimensional theory by rewriting C_{μ} as $C_{\mu} = \frac{1}{2\rho}\epsilon^{\mu\nu\lambda}\partial_{\nu}B_{\lambda}$, where B_{μ} is a new emergent gauge field. Here we note that dimensionality is important as the same condition in 3+1 dimensional theory would be satisfield by curm of a 2-form field whereas in 2+1 dimension the emergent gauge field is 1-form. Also the multivalued part of the scalar χ^s constitute the solinoidal part of the superfluid velocity and the curl of it, under relativistic generalization, is given by

$$\epsilon^{\mu\nu\lambda}\partial_{\nu}\partial_{\lambda}\theta^s = 2\pi n\int d\tau v^{\mu}\delta^3(x - x(\tau)), \quad v^{\mu} = \frac{dx^{\mu}}{d\tau}. \tag{41}$$

We can recognize that the right hand side of Eq. (41) represents a world line current of a particle with position $x_\mu(\tau)$ parametrized by τ. Here the core of the vortex acts as a particle. With this the dual theory can be written as following

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu\rho\partial^\mu\rho + \frac{1}{4\rho^2}H^{\rho\lambda}H_{\rho\lambda} - B_\mu\Sigma^\mu - \frac{q}{2}\varepsilon^{\mu\nu\rho}B_\mu F_{\nu\rho} + V(\rho^2) \\ &+ \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - eA_\mu\bar{\psi}\gamma^\mu\psi\,,\end{aligned} \tag{42}$$

where we have defined $H^{\rho\lambda}$ to be field strength of the new emergent gauge field B_μ ans Σ_μ as the vorticity or the world line current of the core point as expressed by Eq. (41). We shall now proceed to diagonalize the action be removing $B \wedge F$ term. For that one need to linearize the $F_{\mu\nu}F^{\mu\nu}$ term by introducing another Gaussian term of the form

$$\int \mathscr{D}\chi_\mu \exp\left(-\frac{i}{4}\int d^3x \left\{\chi_\mu\chi^\mu - \varepsilon^{\nu\rho\lambda}\chi_\nu F_{\rho\lambda} + \frac{1}{4}\left(\varepsilon^{\nu\rho\lambda}F_{\rho\lambda}\right)^2\right\}\right) = \mathcal{N}'\,, \tag{43}$$

where χ_μ is the newly introduced decoupling field. This factor when introduced into the generating functional will linearize the $F_{\mu\nu}F^{\mu\nu}$ term and then A_μ will be auxiliary field with no dynamical term. One can now integrate over A_μ and get a delta function constraint like $\delta\left(\frac{1}{2}\varepsilon^{\nu\rho\lambda}\partial_\rho\left(\chi_\lambda - qB^\nu\right) - e\bar{\psi}\gamma^\nu\psi\right)$. This can be satisfied by the following ansatz

$$\chi_\mu = \partial_\mu\eta + qB_\mu + e\varepsilon_{\mu\rho\lambda}\partial^\rho\frac{1}{\Box}\bar{\psi}\gamma^\lambda\psi\,, \tag{44}$$

where we have introduced a new single valued scalar field η for making the ansatz more general. Thus putting this into the action we finally have

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}\partial_\mu\rho\partial^\mu\rho + \frac{1}{4\rho^2}H^{\mu\nu}H_{\mu\nu} - \frac{1}{2}B_\mu\Sigma^\mu + V(\rho^2) + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi \\ &- \frac{1}{2}\left(\partial_\mu\eta + qB_\mu\right)^2 - \frac{1}{2}eqB^\mu\varepsilon_{\mu\rho\lambda}\partial^\rho\frac{1}{\Box}\bar{\psi}\gamma^\lambda\psi - \frac{1}{2}e^2\bar{\psi}\gamma^\lambda\psi\frac{1}{\Box}\bar{\psi}\gamma^\lambda\psi,\end{aligned} \tag{45}$$

where the last term is the coulombic interaction between fermions. This is the dual field theory that we wanted to derive.

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